01204211: Exercises 9-1

Notes: To avoid confusion, in your answer, you can write vectors with this notation $\vec{u}$.

1. In this problem, we will use Gaussian Elimination to solve the following linear system.

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & 2x_3 & = & 7 \\ 5x_1 & + & 10x_2 & + & x_3 & = & 8 \\ 2x_1 & - & 4x_2 & - & 4x_3 & = & 9 \end{array}$$

   (a) Formulate the linear system in matrix form. Denote the coefficient matrix by A.

   (b) Use elimination process to transform A into an upper triangular matrix. Write down every step. (There should be 3 steps.)

   Also, for each step i, (1) write down an elementary matrix E_i that represents the elimination process that you perform and (2) write down its inverse E_i^{-1}.

(c) Call the resulting upper triangular matrix U. Write down, in full, the matrix equation $E_3E_2E_1A = U$.

(d) Use the inverses found in question (b) to write down another equation that shows how to perform matrix multiplication to obtain A from U, i.e., $A = E_1^{-1}E_2^{-1}E_3^{-1}U$.

(e) Let $L = E_1^{-1}E_2^{-1}E_3^{-1}U$. Find L and write down the LU decomposition of A, i.e., $A = LU$.

(f) Consider the linear system $Ax = b$ when $A = LU$. Write down the starting linear system that we want to solve in the matrix form, but this time replace A with LU. Explain briefly what the two matrices U and L "do" to x to obtain b.

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

2. Suppose that A is a 3×3 matrix. Write down an elementary matrix E that swap the first and the third rows of A.

3. Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^4$ defined as

$$\begin{array}{lcl} \boldsymbol{u}_1 & = & [\ 1,\ 2,\ 0,\ 4\ ] \\ \boldsymbol{u}_2 & = & [\ 2,\ 0,\ 4,\ 0\ ] \\ \boldsymbol{u}_3 & = & [\ 2,\ 2,\ 3,\ 3\ ] \end{array}$$

We would like to show that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are linearly independent. By definition, this means that if we write

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3 = 0,$$

the only values that α_i's can take are $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Formulate this condition as a linear system (with 3 variables x_1, x_2, x_3 for $\alpha_1, \alpha_2, \alpha_3$ and 4 equations).

Use Gaussian elimination to solve the linear system. Note that although there are more equations than variables, you can still proceed with the same elimination procedure. You may need to swap rows during the elimination process.

4. Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^4$ defined as

$$\begin{array}{lcl} \boldsymbol{u}_1 & = & [\ 1,\ \ 2,\ \ 0,\ \ 4\ ] \\ \boldsymbol{u}_2 & = & [\ 2,\ \ 0,\ \ 4,\ \ 0\ ] \\ \boldsymbol{u}_3 & = & [\ 4,\ \ 4,\ \ 4,\ \ 8\ ] \end{array}$$

Follow the same procedure as in the previous question. What do you see after the Gaussian elimination process is done? Is the set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ linearly independent?