Skeletons

CSE169: Computer Animation

Instructor: Steve Rotenberg

UCSD, Winter 2005

Kinematics

- Kinematics: The analysis of motion independent of physical forces. Kinematics deals with position, velocity, acceleration, and their rotational counterparts, orientation, angular velocity, and angular acceleration.
- Forward Kinematics: The process of computing world space geometric data from DOFs
- Inverse Kinematics: The process of computing a set of DOFs that causes some world space goal to be met (I.e., place the hand on the door knob…)
- Note: Kinematics is an entire branch of mathematics and there are several other aspects of kinematics that don’t fall into the ‘forward’ or ‘inverse’ description

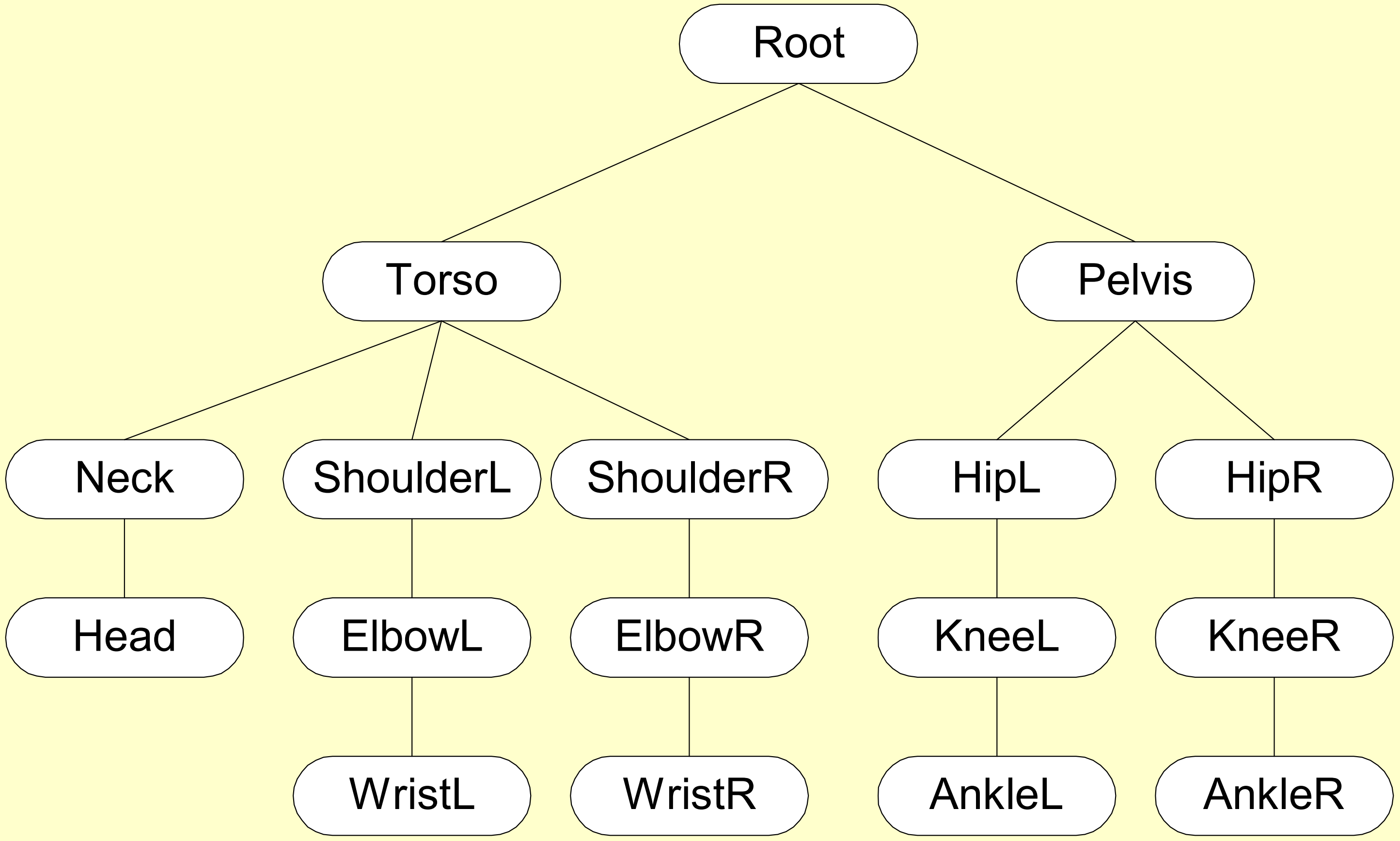
Skeletons

- Skeleton: A pose-able framework of joints arranged in a tree structure. The skeleton is used as an invisible armature to manipulate the skin and other geometric data of the character
- Joint: A joint allows relative movement within the skeleton. Joints are essentially 4x4 matrix transformations. Joints can be rotational, translational, or some non-realistic types as well
- Bone: Bone is really just a synonym for joint for the most part. For example, one might refer to the shoulder joint or upper arm bone (humerus) and mean the same thing

DOFs

- Degree of Freedom (DOF): A variable φ describing a particular axis or dimension of movement within a joint
- Joints typically have around 1-6 DOFs ($\varphi_1 \ldots \varphi_N$)
- Changing the DOF values over time results in the animation of the skeleton
- In later weeks, we will extend the concept of a DOF to be any animatable parameter within the character rig
- Note: in a mathematical sense, a free rigid body has 6 DOFs: 3 for position and 3 for rotation

Example Joint Hierarchy



Joints

- Core Joint Data
  - DOFs (N floats)
  - Local matrix: **L**
  - World matrix: **W**
- Additional Data
  - Joint offset vector: **r**
  - DOF limits (min & max value per DOF)
  - Type-specific data (rotation/translation axes, constants…)
  - Tree data (pointers to children, siblings, parent…)

Skeleton Posing Process

1. Specify all DOF values for the skeleton (done by higher level animation system)

2. Recursively traverse through the hierarchy starting at the root and use forward kinematics to compute the world matrices (done by skeleton system)

3. Use world matrices to deform skin & render (done by skin system)

Note: the matrices can also be used for other things such as collision detection, FX, etc.

Forward Kinematics

- In the recursive tree traversal, each joint first computes its local matrix **L** based on the values of its DOFs and some formula representative of the joint type:

  Local matrix $\mathbf{L} = \mathbf{L}_{joint}(\varphi_1,\varphi_2,\ldots,\varphi_N)$

- Then, world matrix **W** is computed by concatenating **L** with the world matrix of the parent joint

  World matrix $\mathbf{W} = \mathbf{L} \cdot \mathbf{W}_{parent}$

Joint Offsets

- It is convenient to have a 3D offset vector **r** for every joint which represents its pivot point relative to its parent's matrix

$$\mathbf{L}_{offset} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

DOF Limits

- It is nice to be able to limit a DOF to some range (for example, the elbow could be limited from 0º to 150º)
- Usually, in a realistic character, all DOFs will be limited except the ones controlling the root

Skeleton Rigging

- Setting up the skeleton is an important and early part of the rigging process
- Sometimes, character skeletons are built before the skin, while other times, it is the opposite
- To set up a skeleton, an artist uses an interactive tool to:
  - Construct the tree
  - Place joint offsets
  - Configure joint types
  - Specify joint limits
  - Possibly more…

Poses

- Once the skeleton is set up, one can then adjust each of the DOFs to specify the pose of the skeleton
- We can define a pose Φ more formally as a vector of N numbers that maps to a set of DOFs in the skeleton

$$\Phi = [\varphi_1\ \varphi_2\ \ldots\ \varphi_N]$$

- A pose is a convenient unit that can be manipulated by a higher level animation system and then handed down to the skeleton
- Usually, each joint will have around 1-6 DOFs, but an entire character might have 100+ DOFs in the skeleton
- Keep in mind that DOFs can be also used for things other than joints, as we will learn later…

Joint Types

Joint Types

- Rotational
  - Hinge: 1-DOF
  - Universal: 2-DOF
  - Ball & Socket: 3-DOF
    - Euler Angles
    - Quaternions
- Translational
  - Prismatic: 1-DOF
  - Translational: 3-DOF (or any number)
- Compound
  - Free
  - Screw
  - Constraint
  - Etc.
- Non-Rigid
  - Scale
  - Shear
  - Etc.
- Design your own...

Hinge Joints (1-DOF Rotational)

- Rotation around the x-axis:

$$\mathbf{L}_{Rx}(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ 0 & -\sin\theta_x & \cos\theta_x & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

Hinge Joints (1-DOF Rotational)

- Rotation around the y-axis:

$$\mathbf{L}_{Ry}\left(\theta_y\right)=\begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

Hinge Joints (1-DOF Rotational)

- Rotation around the z-axis:

$$\mathbf{L}_{Rz}(\theta_z) = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 & 0 \\ -\sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

Hinge Joints (1-DOF Rotational)

- Rotation around an arbitrary axis **a**:

$$\mathbf{L}_{Ra}(\theta) =$$

$$\begin{bmatrix} a_x^2 + c_\theta(1-a_x^2) & a_x a_y(1-c_\theta)+a_z s_\theta & a_x a_z(1-c_\theta)-a_y s_\theta & 0 \\ a_x a_y(1-c_\theta)-a_z s_\theta & a_y^2 + c_\theta(1-a_y^2) & a_y a_z(1-c_\theta)+a_x s_\theta & 0 \\ a_x a_z(1-c_\theta)+a_y s_\theta & a_y a_z(1-c_\theta)-a_x s_\theta & a_z^2 + c_\theta(1-a_z^2) & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

Universal Joints (2-DOF)

- For a 2-DOF joint that first rotates around x and then around y:

$$\mathbf{L}_{Rxy}\left(\theta_x,\theta_y\right)=\begin{bmatrix} c_y & 0 & -s_y & 0 \\ s_x s_y & c_x & s_x c_y & 0 \\ c_x s_y & -s_x & c_x c_y & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

- Different matrices can be formed for different axis combinations

Ball & Socket (3-DOF)

- For a 3-DOF joint that first rotates around x, y, then z:

$$\mathbf{L}_{Rxyz}\left(\theta_x,\theta_y,\theta_z\right)=\begin{bmatrix} c_y c_z & c_y s_z & -s_y & 0 \\ s_x s_y c_z - c_x s_z & s_x s_y s_z + c_x c_z & s_x c_y & 0 \\ c_x s_y c_z + s_x s_z & c_x s_y s_z - s_x c_z & c_x c_y & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

- Different matrices can be formed for different axis combinations

Quaternions

$$\mathbf{q} = \begin{bmatrix} q_x & q_y & q_z & q_w \end{bmatrix}$$

$$|\mathbf{q}| = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2} = 1$$

$$\mathbf{q} = \begin{bmatrix} a_x \sin\frac{\theta}{2} & a_y \sin\frac{\theta}{2} & a_z \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$\mathbf{L}_Q(\mathbf{q}) = \begin{bmatrix} 1-2q_y^2-2q_z^2 & 2q_xq_y+2q_wq_z & 2q_xq_z-2q_wq_y & 0 \\ 2q_xq_y-2q_wq_z & 1-2q_x^2-2q_z^2 & 2q_yq_z+2q_wq_x & 0 \\ 2q_xq_z+2q_wq_y & 2q_yq_z-2q_wq_x & 1-2q_x^2-2q_y^2 & 0 \\ r_x & r_y & r_z & 1 \end{bmatrix}$$

Prismatic Joints (1-DOF Translation)

- 1-DOF translation along an arbitrary axis **a**:

$$\mathbf{L}_{Ta}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r_x + t \cdot a_x & r_y + t \cdot a_y & r_z + t \cdot a_z & 1 \end{bmatrix}$$

Translational Joints (3-DOF)

- For a more general 3-DOF translation:

$$\mathbf{L}_{Txyz}(\mathbf{t}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r_x + t_x & r_y + t_y & r_z + t_z & 1 \end{bmatrix}$$

Other Joints

- Compound
  - Free
  - Screw
  - Constraint
  - Etc.
- Non-Rigid
  - Scale (1 axis, 3 axis, volume preserving…)
  - Shear
  - Etc.

Skin

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Texture

- We may wish to ‘map’ various properties across the polygonal surface
- We can do this through texture mapping, or other more general mapping techniques
- Usually, this will require explicitly storing texture coordinate information at the vertices
- For higher quality rendering, we may combine several different maps in complex ways, each with their own mapping coordinates
- Related features include bump mapping, displacement mapping, illumination mapping…

Smooth Skin Algorithm

Weighted Blending & Averaging

- Weighted sum: $x' = \sum_{i=0} w_i x_i$

- Weighted average: $\sum_{i=0} w_i = 1$

- Convex average: $0 \leq w_i \leq 1$

Rigid Parts

- Robots and mechanical creatures can usually be rendered with rigid parts and don’t require a smooth skin
- To render rigid parts, each part is transformed by its joint matrix independently
- In this situation, every vertex of the character’s geometry is transformed by exactly one matrix

$$\mathbf{v}' = \mathbf{v} \cdot \mathbf{W}$$

where v is defined in joint’s local space

Simple Skin

- A simple improvement for low-medium quality characters is to rigidly bind a skin to the skeleton. This means that every vertex of the continuous skin mesh is attached to a joint.
- In this method, as with rigid parts, every vertex is transformed exactly once and should therefore have similar performance to rendering with rigid parts.

$$\mathbf{v}' = \mathbf{v} \cdot \mathbf{W}$$

Smooth Skin

- With the smooth skin algorithm, a vertex can be attached to more than one joint with adjustable weights that control how much each joint affects it
- Verts rarely need to be attached to more than three joints
- Each vertex is transformed a few times and the results are blended
- The smooth skin algorithm has many other names: blended skin, skeletal subspace deformation (SSD), multi-matrix skin, matrix palette skinning…

Smooth Skin Algorithm

- The deformed vertex position is a weighted average:

$$\mathbf{v}' = w_1(\mathbf{v}\cdot\mathbf{M}_1) + w_2(\mathbf{v}\cdot\mathbf{M}_2) + \ldots w_N(\mathbf{v}\cdot\mathbf{M}_N)$$

or

$$\mathbf{v}' = \sum w_i(\mathbf{v}\cdot\mathbf{M}_i)$$

where

$$\sum w_i = 1$$

Binding Matrices

- With rigid parts or simple skin, **v** can be defined local to the joint that transforms it
- With smooth skin, several joints transform a vertex, but it can't be defined local to all of them
- Instead, we must first transform it to be local to the joint that will then transform it to the world
- To do this, we use a binding matrix **B** for each joint that defines where the joint was when the skin was attached and premultiply its inverse with the world matrix:

$$\mathbf{M}_i = \mathbf{B}_i^{-1} \cdot \mathbf{W}_i$$

Normals

- To compute shading, we need to transform the normals to world space also
- Because the normal is a direction vector, we don't want it to get the translation from the matrix, so we only need to multiply the normal by the upper 3x3 portion of the matrix
- For a normal bound to only one joint:

$$\mathbf{n}' = \mathbf{n} \cdot \mathbf{W}$$

Normals

- For smooth skin, we must blend the normal as with the positions, but the normal must then be renormalized:

$$\mathbf{n}' = \frac{\sum w_i \left(\mathbf{n} \cdot \mathbf{M}_i\right)}{\left|\sum w_i \left(\mathbf{n} \cdot \mathbf{M}_i\right)\right|}$$

- If the matrices have non-rigid transformations, then technically, we should use:

$$\mathbf{n}' = \frac{\sum w_i \left(\mathbf{n} \cdot \mathbf{M}_i^{-1T}\right)}{\left|\sum w_i \left(\mathbf{n} \cdot \mathbf{M}_i^{-1T}\right)\right|}$$

Algorithm Overview

Skin::Update() (view independent processing)

- Compute skinning matrix for each joint: $\mathbf{M}=\mathbf{B}^{-1}\cdot\mathbf{W}$ (you can precompute and store $\mathbf{B}^{-1}$ instead of **B**)
- Loop through vertices and compute blended position & normal

Skin::Draw() (view dependent processing)

- Set matrix state to Identity (world)
- Loop through triangles and draw using world space positions & normals

Questions:

- Why not deal with **B** in Skeleton::Update() ?
- Why not just transform vertices within Skin::Draw() ?

Rig Data Flow

- Input DOFs
- Rigging system (skeleton, skin…)
- Output renderable mesh (vertices, normals…)

$$\begin{tikzcd}
\mathbf{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_N \end{bmatrix} \arrow[d, Rightarrow] \\
\boxed{\text{Rig}} \arrow[d, Rightarrow] \\
\mathbf{v}', \mathbf{n}'
\end{tikzcd}$$

Skeleton Forward Kinematics

- Every joint computes a local matrix based on its DOFs and any other constants necessary (joint offsets…)

$$\mathbf{L} = \mathbf{L}_{jnt}\left(\phi_1, \phi_2, \ldots, \phi_N\right)$$

- To find the joint's world matrix, we compute the dot product of the local matrix with the parent's world matrix

$$\mathbf{W} = \mathbf{L} \cdot \mathbf{W}_{parent}$$

- Normally, we would do this in a depth-first order starting from the root, so that we can be sure that the parent's world matrix is available when its needed

Smooth Skin Algorithm

- The deformed vertex position is a weighted average over all of the joints that the vertex is attached to:

$$\mathbf{v}' = \sum w_i \mathbf{v} \cdot \mathbf{B}_i^{-1} \cdot \mathbf{W}_i$$

- **W** is a joint's world matrix and **B** is a joint's binding matrix that describes where it's world matrix was when it was attached to the skin model (at skin creation time)
- Each joint transforms the vertex as if it were rigidly attached, and then those results are blended based on user specified weights
- All of the weights must add up to 1: $\sum w_i = 1$
- Blending normals is essentially the same, except we transform them as direction vectors (x,y,z,0) and then renormalize the results

$$\mathbf{n}^* = \sum w_i \mathbf{n} \cdot \mathbf{B}_i^{-1} \cdot \mathbf{W}_i, \qquad \mathbf{n}' = \frac{\mathbf{n}^*}{\left|\mathbf{n}^*\right|}$$

Skinning Equations

- Skeleton

$$\mathbf{L} = \mathbf{L}_{jnt}\left(\phi_1, \phi_2, \ldots, \phi_N\right)$$

$$\mathbf{W} = \mathbf{L} \cdot \mathbf{W}_{parent}$$

- Skinning

$$\mathbf{v}' = \sum w_i \mathbf{v} \cdot \mathbf{B}_i^{-1} \cdot \mathbf{W}_i$$

$$\mathbf{n}^* = \sum w_i \mathbf{n} \cdot \mathbf{B}_i^{-1} \cdot \mathbf{W}_i$$

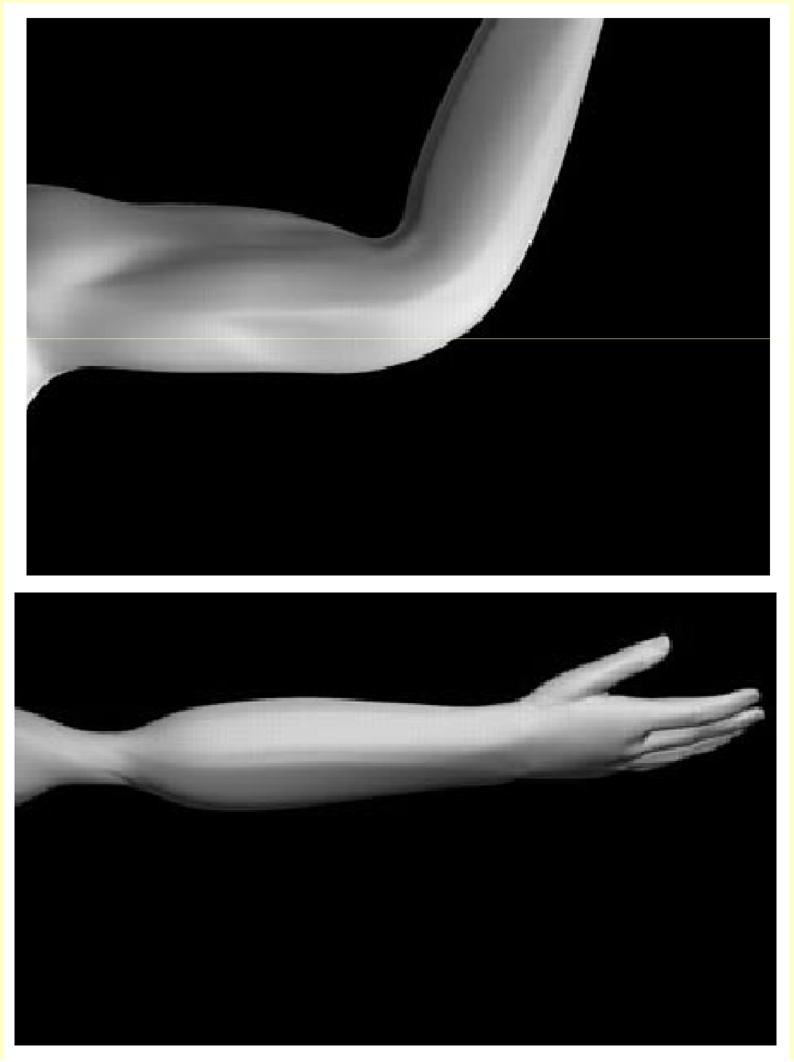
$$\mathbf{n}' = \frac{\mathbf{n}^*}{\left|\mathbf{n}^*\right|}$$

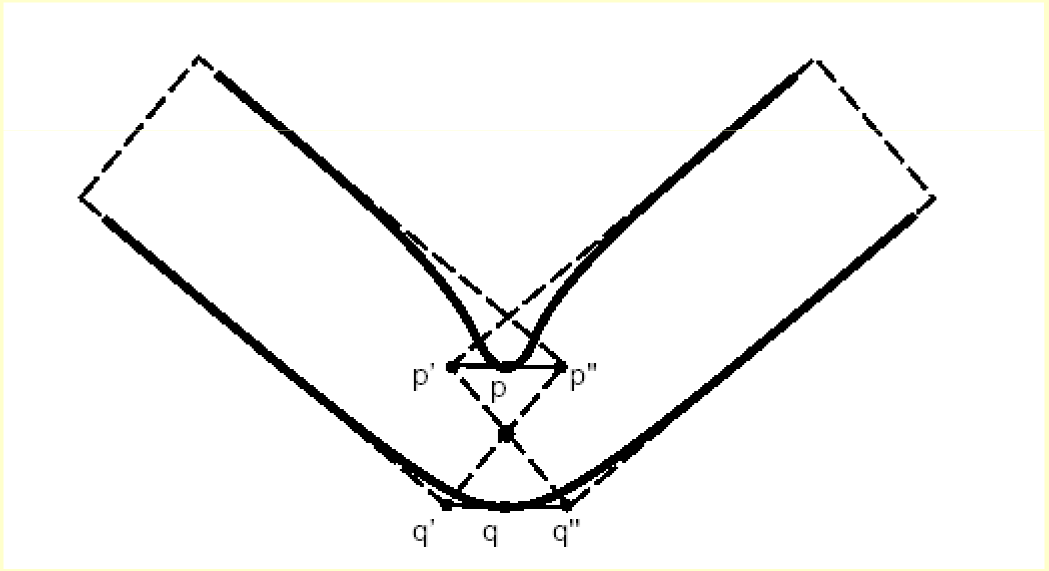
Using Skinning

Limitations of Smooth Skin

- Smooth skin is very simple and quite fast, but its quality is limited
- The main problems are:
  - Joints tend to collapse as they bend more
  - Very difficult to get specific control
  - Unintuitive and difficult to edit
- Still, it is built in to most 3D animation packages and has support in both OpenGL and Direct3D
- If nothing else, it is a good baseline upon which more complex schemes can be built

Limitations of Smooth Skin





Bone Links

- To help with the collapsing joint problem, one option is to use bone links
- Bone links are extra joints inserted in the skeleton to assist with the skinning
- They can be automatically added based on the joint’s range of motion. For example, they could be added so as to prevent any joint from rotating more than 60 degrees.
- This is a simple approach used in some real time games, but doesn’t go very far in fixing the other problems with smooth skin.

Shape Interpolation

- Another extension to the smooth skinning algorithm is to allow the verts to be modeled at key values along the joints motion
- For an elbow, for example, one could model it straight, then model it fully bent
- These shapes are interpolated local to the bones before the skinning is applied
- We will talk more about this technique in the next lecture

Muscles & Other Effects

- One can add custom effects such as muscle bulges as additional joints
- For example, the bicep could be a translational or scaling joint that smoothly controls some of the verts in the upper arm. Its motion could be linked to the motion of the elbow rotation.
- With this approach, one can also use skin for muscles, fat bulges, facial expressions, and even simple clothing
- We will learn more about advanced skinning techniques in a later lecture

Rigging Process

- To rig a skinned character, one must have a geometric skin mesh and a skeleton
- Usually, the skin is built in a relatively neutral pose, often in a comfortable standing pose
- The skeleton, however, might be built in more of a *zero pose* where the joints DOFs are assumed to be 0, causing a very stiff, straight pose
- To attach the skin to the skeleton, the skeleton must first be posed into a *binding pose*
- Once this is done, the verts can be assigned to joints with appropriate weights

Skin Binding

- Attaching a skin to a skeleton is not a trivial problem and usually requires automated tools combined with extensive interactive tuning
- Binding algorithms typically involve heuristic approaches
- Some general approaches:
  - Containment
  - Point-to-line mapping
  - Delaunay tetrahedralization

Containment Binding

- With containment binding algorithms, the user manually approximates the body with volume primitives for each bone (cylinders, ellipsoids, spheres…)
- The algorithm then tests each vertex against the volumes and attaches it to the best fitting bone
- Some containment algorithms attach to only one bone and then use smoothing as a second pass. Others attach to multiple bones directly and set skin weights
- For a more automated version, the volumes could be initially set based on the bone lengths and child locations

Point-to-Line Mapping

- A simple way to attach a skin is treat each bone as one or more line segments and attach each vertex to the nearest line segment
- A bone is made from line segments connecting the joint pivot to the pivots of each child

Delaunay Tetrahedralization

- This tricky computational geometry technique builds a tetrahedralization of the volume within the skin
- The tetrahedra connect all of the skin verts and skeletal pivots in a relatively clean ‘Delaunay’ fashion
- The connectivity of the mesh can then be analyzed to determine the best attachment for each vertex

Skin Adjustment

- Mesh Smoothing: A joint will first be attached in a fairly rigid fashion (either automatic or manually) and then the weights are smoothed algorithmically
- Rogue Removal: Automatic identification and removal of isolated vertex attachments
- Weight Painting: Some 3D tools allow visualization of the weights as colors (0…1 -> black…white). These can then be adjusted and 'painted' in an interactive fashion
- Direct Manipulation: These algorithms allow the vertex to be moved to a 'correct' position after the bone is bent, and automatically compute the weights necessary to get it there

Hardware Skinning

- The smooth skinning algorithm is simple and popular enough to have some direct support in 3D rendering hardware
- Actually, it just requires standard vector multiply/add operations and so can be implemented in microcode

Skin Memory Usage

- For each vertex, we need to store:
  - Rendering data (position, normal, color, texture coords, tangents…)
  - Skinning data (number of attachments, joint index, weight…)
- If we limit the character to having at most 256 bones, we can store a bone index as a byte
- If we limit the weights to 256 distinct values, we can store a weight as a byte (this gives us a precision of 0.004%, which is fine)
- If we assume that a vertex will attach to at most 4 bones, then we can compress the skinning data to (1+1)*4 =8 bytes per vertex (64 bits)
- In fact, we can even squeeze another 8 bits out of that by not storing the final weight, since

$$w3 = 1 - w0 - w1 - w2$$

Inverse Kinematics (part 1)

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Welman, 1993

- “Inverse Kinematics and Geometric Constraints for Articulated Figure Manipulation”, Chris Welman, 1993
- Masters thesis on IK algorithms
- Examines Jacobian methods and Cyclic Coordinate Descent (CCD)
- Please read sections 1-4 (about 40 pages)

Forward Kinematics

- The local and world matrix construction within the skeleton is an implementation of *forward kinematics*
- Forward kinematics refers to the process of computing world space geometric descriptions (matrices…) based on joint DOF values (usually rotation angles and/or translations)

Kinematic Chains

- For today, we will limit our study to linear kinematic chains, rather than the more general hierarchies (i.e., stick with individual arms & legs rather than an entire body with multiple branching chains)

End Effector

- The joint at the root of the chain is sometimes called the *base*
- The joint (bone) at the leaf end of the chain is called the *end effector*
- Sometimes, we will refer to the end effector as being a bone with position and orientation, while other times, we might just consider a point on the tip of the bone and only think about it’s position

Forward Kinematics

- We will use the vector:

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_M \end{bmatrix}$$

  to represent the array of M joint DOF values
- We will also use the vector:

$$\mathbf{e} = \begin{bmatrix} e_1 & e_2 & \ldots & e_N \end{bmatrix}$$

  to represent an array of N DOFs that describe the end effector in world space. For example, if our end effector is a full joint with orientation, **e** would contain 6 DOFs: 3 translations and 3 rotations. If we were only concerned with the end effector position, **e** would just contain the 3 translations.

Forward Kinematics

- The forward kinematic function f() computes the world space end effector DOFs from the joint DOFs:

$$\mathbf{e} = f\left(\mathbf{\Phi}\right)$$

Inverse Kinematics

- The goal of inverse kinematics is to compute the vector of joint DOFs that will cause the end effector to reach some desired goal state
- In other words, it is the inverse of the forward kinematics problem

$$\mathbf{\Phi} = f^{-1}(\mathbf{e})$$

Inverse Kinematics Issues

- IK is challenging because while f() may be relatively easy to evaluate, $f^{-1}()$ usually isn't
- For one thing, there may be several possible solutions for **Φ**, or there may be no solutions
- Even if there is a solution, it may require complex and expensive computations to find it
- As a result, there are many different approaches to solving IK problems

Analytical vs. Numerical Solutions

- One major way to classify IK solutions is into analytical and numerical methods
- Analytical methods attempt to mathematically solve an exact solution by directly inverting the forward kinematics equations. This is only possible on relatively simple chains.
- Numerical methods use approximation and iteration to converge on a solution. They tend to be more expensive, but far more general purpose.
- Today, we will examine a numerical IK technique based on Jacobian matrices

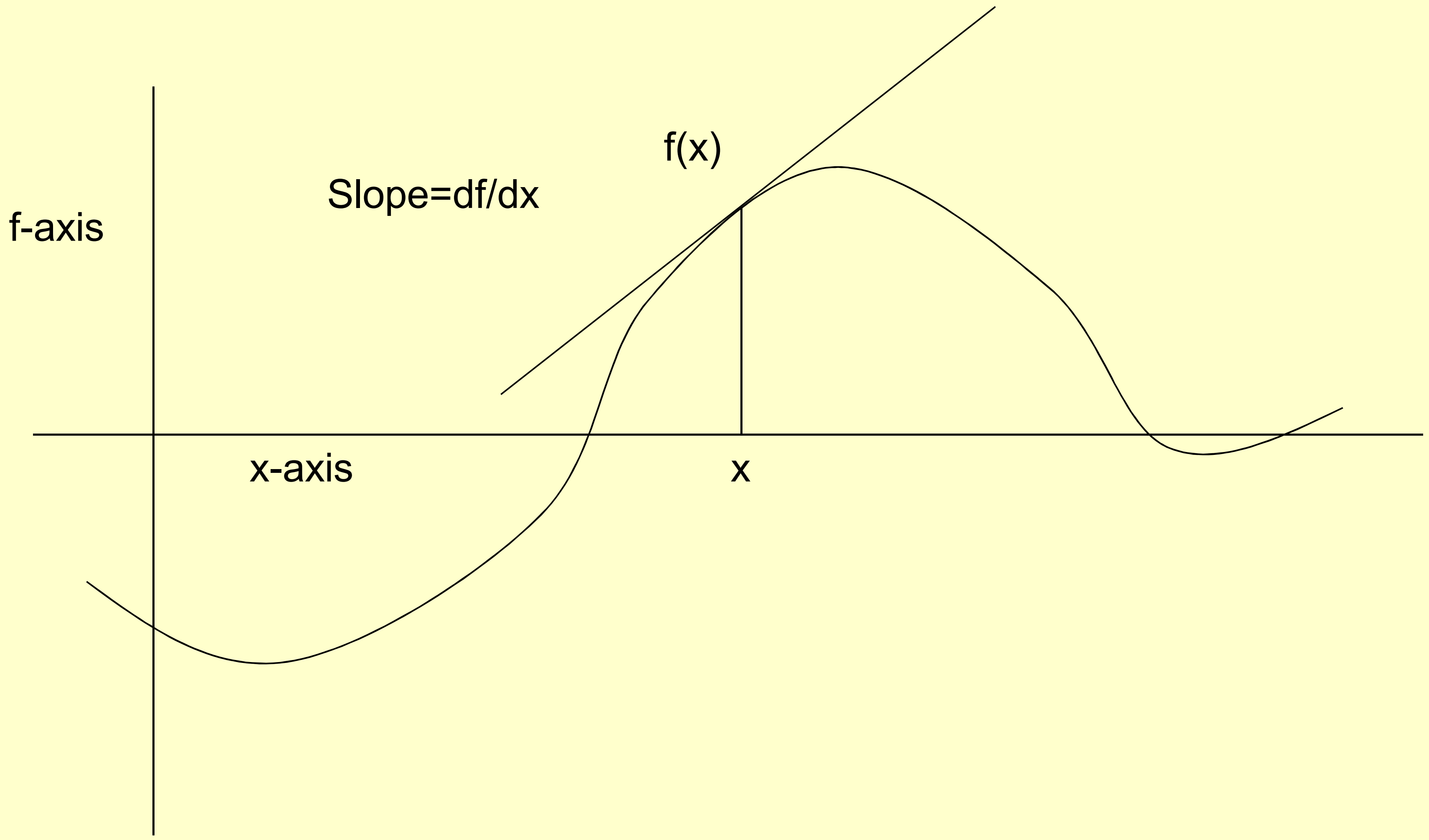
Calculus Review

Derivative of a Scalar Function

- If we have a scalar function f of a single variable x, we can write it as f(x)
- The derivative of the function with respect to x is df/dx
- The derivative is defined as:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Derivative of a Scalar Function



Derivative of f(x)=x^2

For example: $f(x) = x^2$

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} \qquad = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = \boxed{2x}$$

Exact vs. Approximate

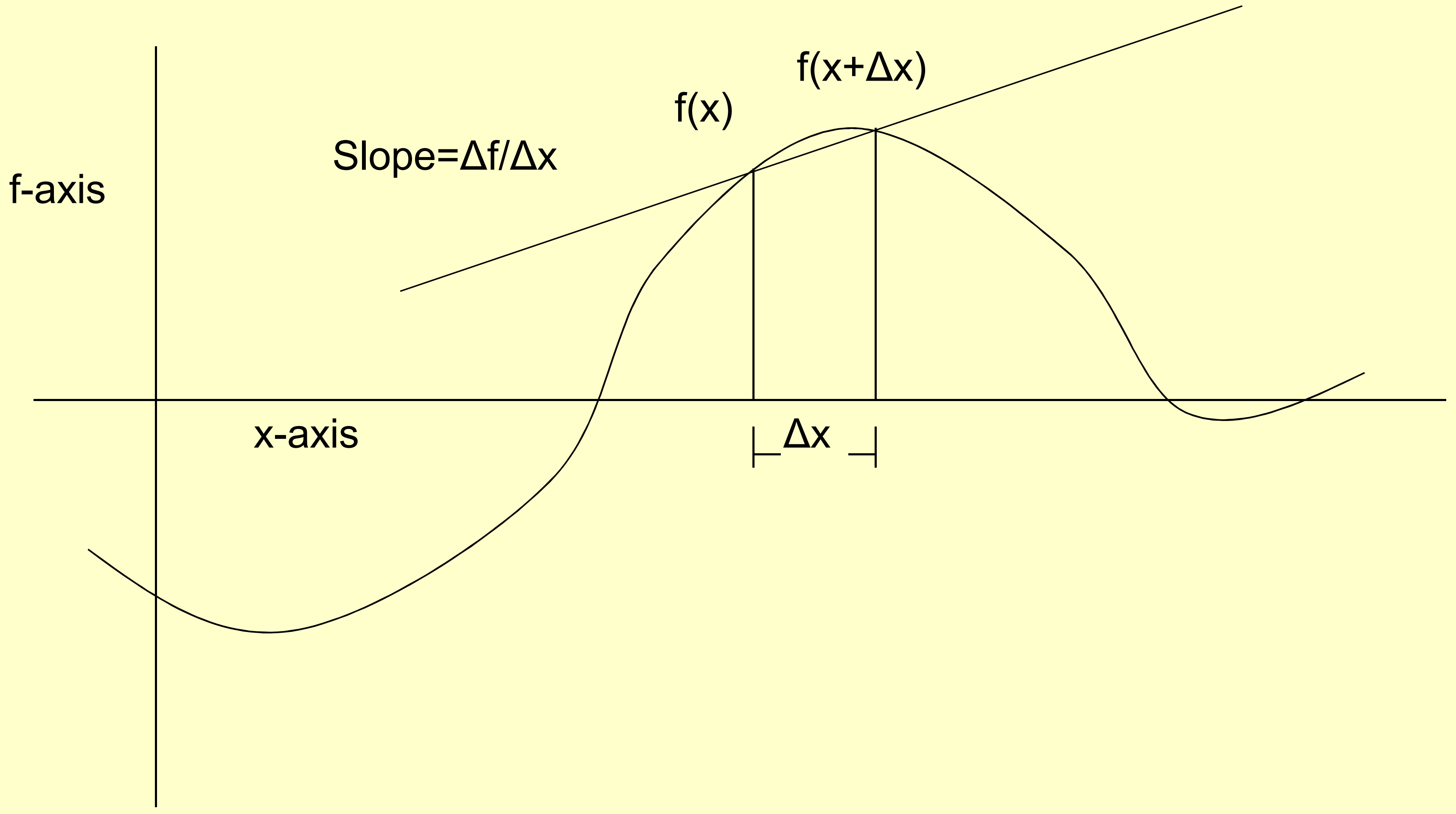
- Many algorithms require the computation of derivatives
- Sometimes, we can compute analytical derivatives. For example:

$$f(x) = x^2 \qquad \frac{df}{dx} = 2x$$

- Other times, we have a function that’s too complex, and we can’t compute an exact derivative
- As long as we can evaluate the function, we can always approximate a derivative

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{for small } \Delta x$$

Approximate Derivative



Nearby Function Values

- If we know the value of a function and its derivative at some x, we can estimate what the value of the function is at other points near x

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx}$$

$$\Delta f \approx \Delta x \frac{df}{dx}$$

$$f\left(x + \Delta x\right) \approx f\left(x\right) + \Delta x \frac{df}{dx}$$

Finding Solutions to $f(x)=0$

- There are many mathematical and computational approaches to finding values of x for which $f(x)=0$
- One such way is the *gradient descent* method
- If we can evaluate $f(x)$ and df/dx for any value of x, we can always follow the gradient (slope) in the direction towards 0

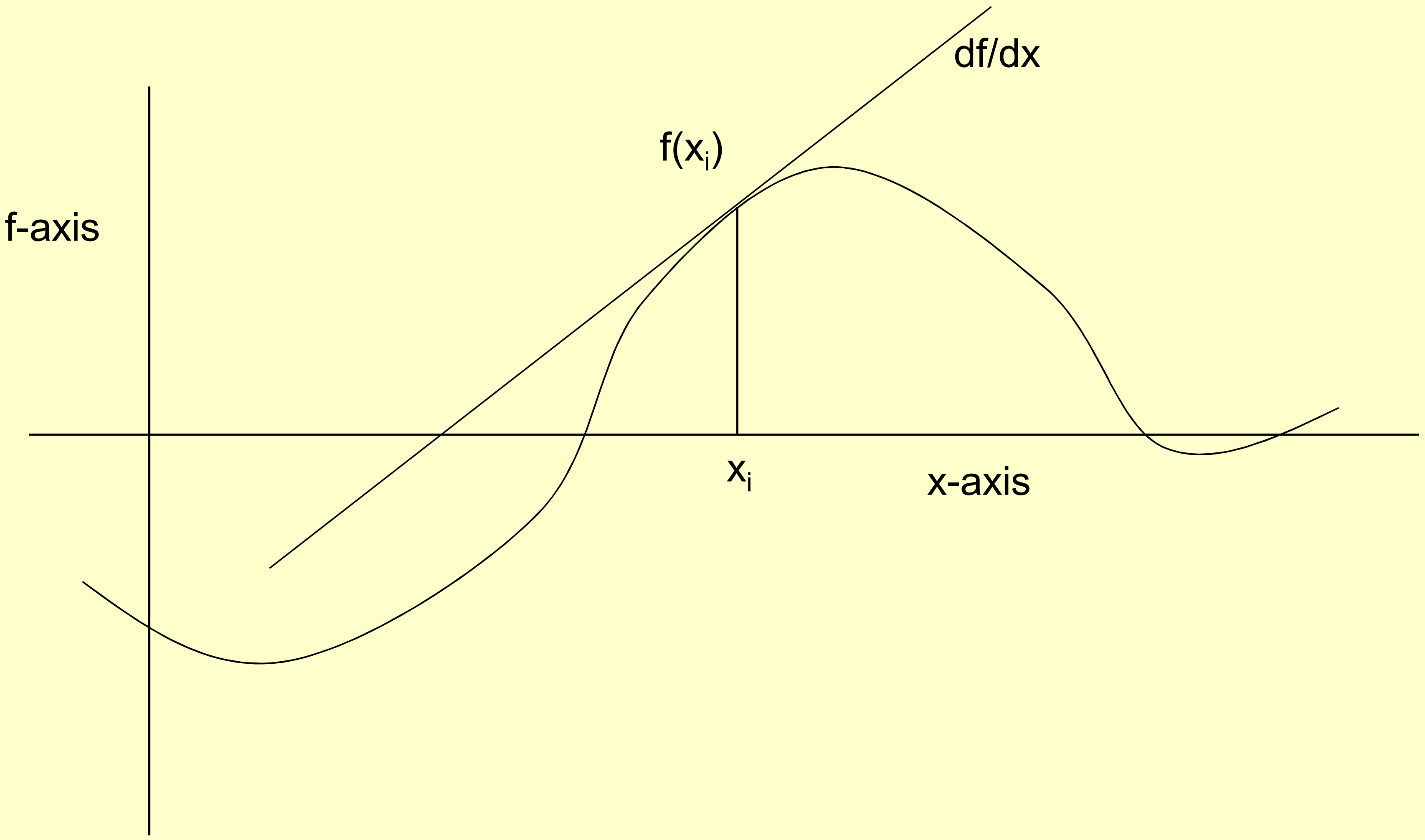
Gradient Descent

- We want to find the value of x that causes f(x) to equal 0
- We will start at some value x_0 and keep taking small steps:

  $$x_{i+1} = x_i + \Delta x$$

  until we find a value x_N that satisfies $f(x_N)=0$
- For each step, we try to choose a value of Δx that will bring us closer to our goal
- We can use the derivative as an approximation to the slope of the function and use this information to move ‘downhill’ towards zero

Gradient Descent



Minimization

- If $f(x_i)$ is not 0, the value of $f(x_i)$ can be thought of as an error. The goal of gradient descent is to minimize this error, and so we can refer to it as a *minimization* algorithm
- Each step Δx we take results in the function changing its value. We will call this change Δf.
- Ideally, we could have $\Delta f = -f(x_i)$. In other words, we want to take a step Δx that causes Δf to cancel out the error
- More realistically, we will just hope that each step will bring us closer, and we can eventually stop when we get 'close enough'
- This iterative process involving approximations is consistent with many *numerical* algorithms

Choosing Δx Step

- If we have a function that varies heavily, we will be safest taking small steps
- If we have a relatively smooth function, we could try stepping directly to where the linear approximation passes through 0

Choosing Δx Step

- If we want to choose Δx to bring us to the value where the slope passes through 0, we can use:

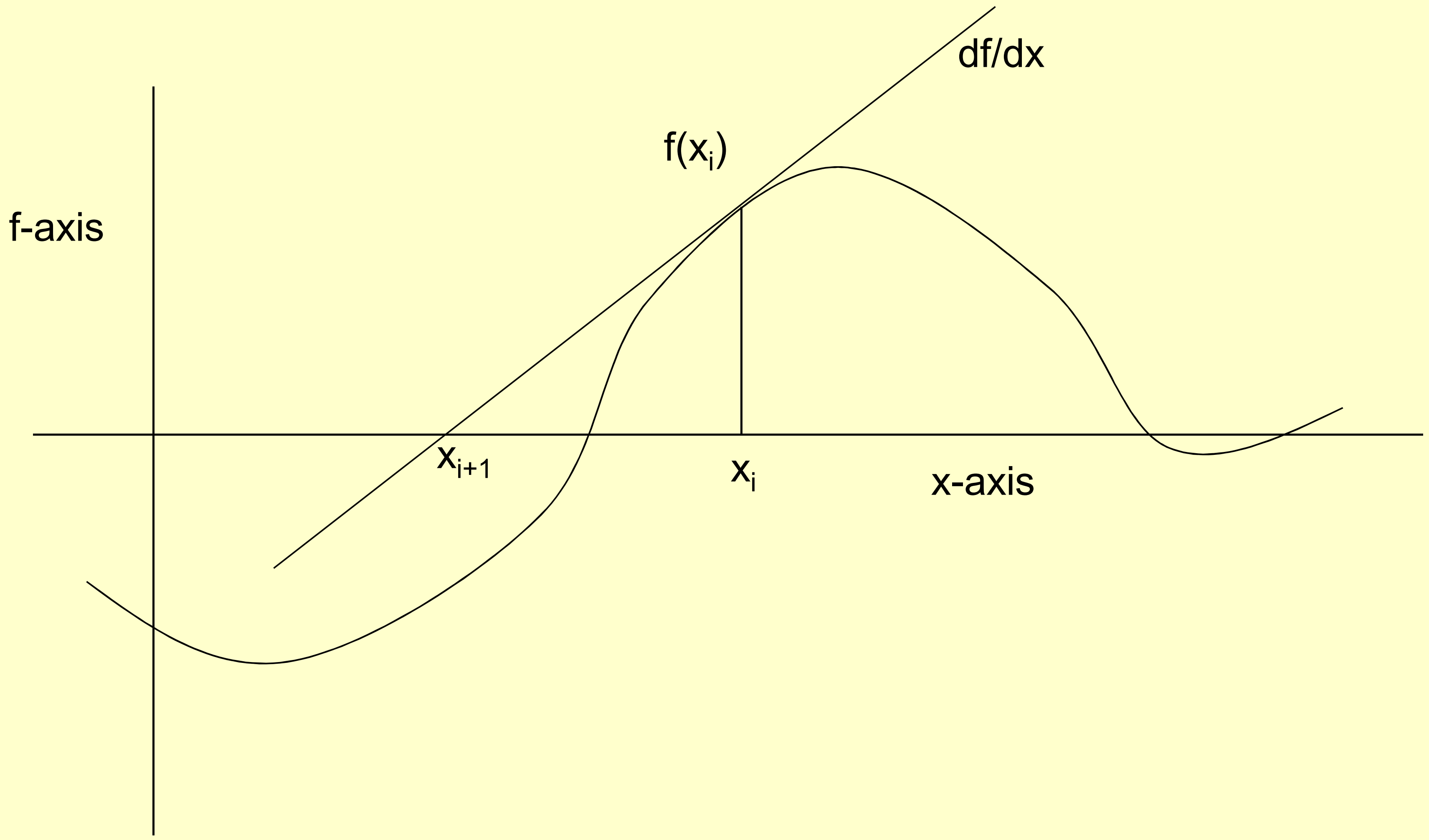
$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx}$$

$$\Delta f \approx \Delta x \frac{df}{dx}$$

$$-f(x_i) \approx \Delta x \frac{df}{dx}$$

$$\Delta x = -f(x_i)\left(\frac{df}{dx}\right)^{-1}$$

Gradient Descent

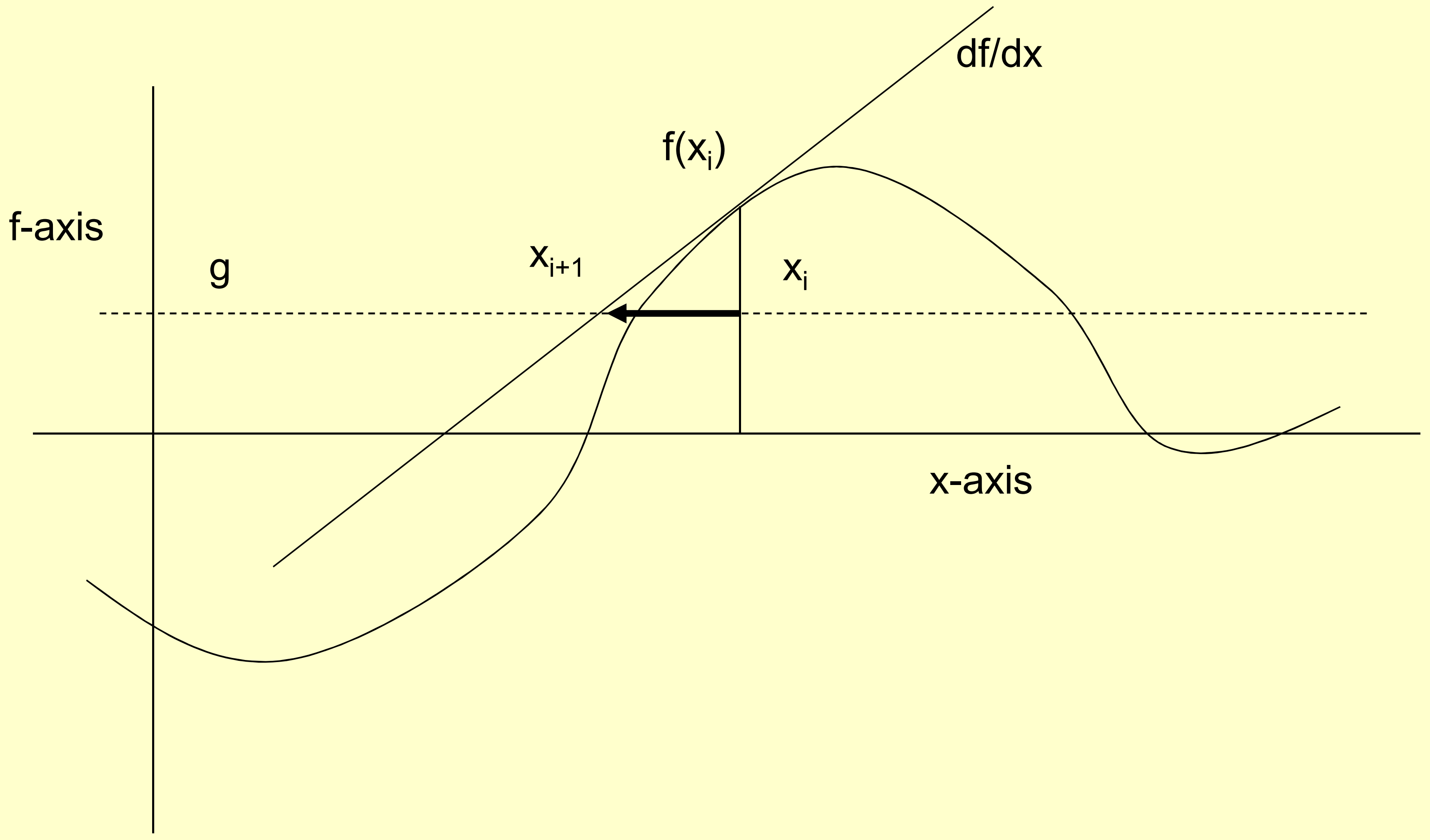


Solving f(x)=g

- If we don’t want to find where a function equals some value ‘g’ other than zero, we can simply think of it as minimizing f(x)-g and just step towards g:

$$\Delta x = \left(g - f\left(x_i\right)\right)\left(\frac{df}{dx}\right)^{-1}$$

Gradient Descent for f(x)=g



Taking Safer Steps

- Sometimes, we are dealing with non-smooth functions with varying derivatives
- Therefore, our simple linear approximation is not very reliable for large values of Δx
- There are many approaches to choosing a more appropriate (smaller) step size
- One simple modification is to add a parameter β to scale our step (0≤ β ≤1)

$$\Delta x = \beta\left(g - f\left(x_i\right)\right)\left(\frac{df}{dx}\right)^{-1}$$

Inverse of the Derivative

- By the way, for *scalar* derivatives:

$$\left(\frac{df}{dx}\right)^{-1} = \frac{1}{\left(\frac{df}{dx}\right)} = \frac{dx}{df}$$

Gradient Descent Algorithm

```
$x_0 = \text{initial starting value}$
$f_0 = f(x_0)$                                   // evaluate $f$ at $x_0$
while $(f_n \neq g)$ {
    $s_i = \frac{df}{dx}(x_i)$                    // compute slope
    $x_{i+1} = x_i + \beta(g - f_i)\frac{1}{s_i}$   // take step along $\Delta x$
    $f_{i+1} = f(x_{i+1})$                        // evaluate $f$ at new $x_{i+1}$
}
```

Stopping the Descent

- At some point, we need to stop iterating
- Ideally, we would stop when we get to our goal
- Realistically, we will stop when we get to within some acceptable tolerance
- However, occasionally, we may get ‘stuck’ in a situation where we can’t make any small step that takes us closer to our goal
- We will discuss some more about this later

Derivative of a Vector Function

- If we have a vector function **r** which represents a particle's position as a function of time t:

$$\mathbf{r} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix}$$

$$\frac{d\mathbf{r}}{dt} = \begin{bmatrix} \frac{dr_x}{dt} & \frac{dr_y}{dt} & \frac{dr_z}{dt} \end{bmatrix}$$

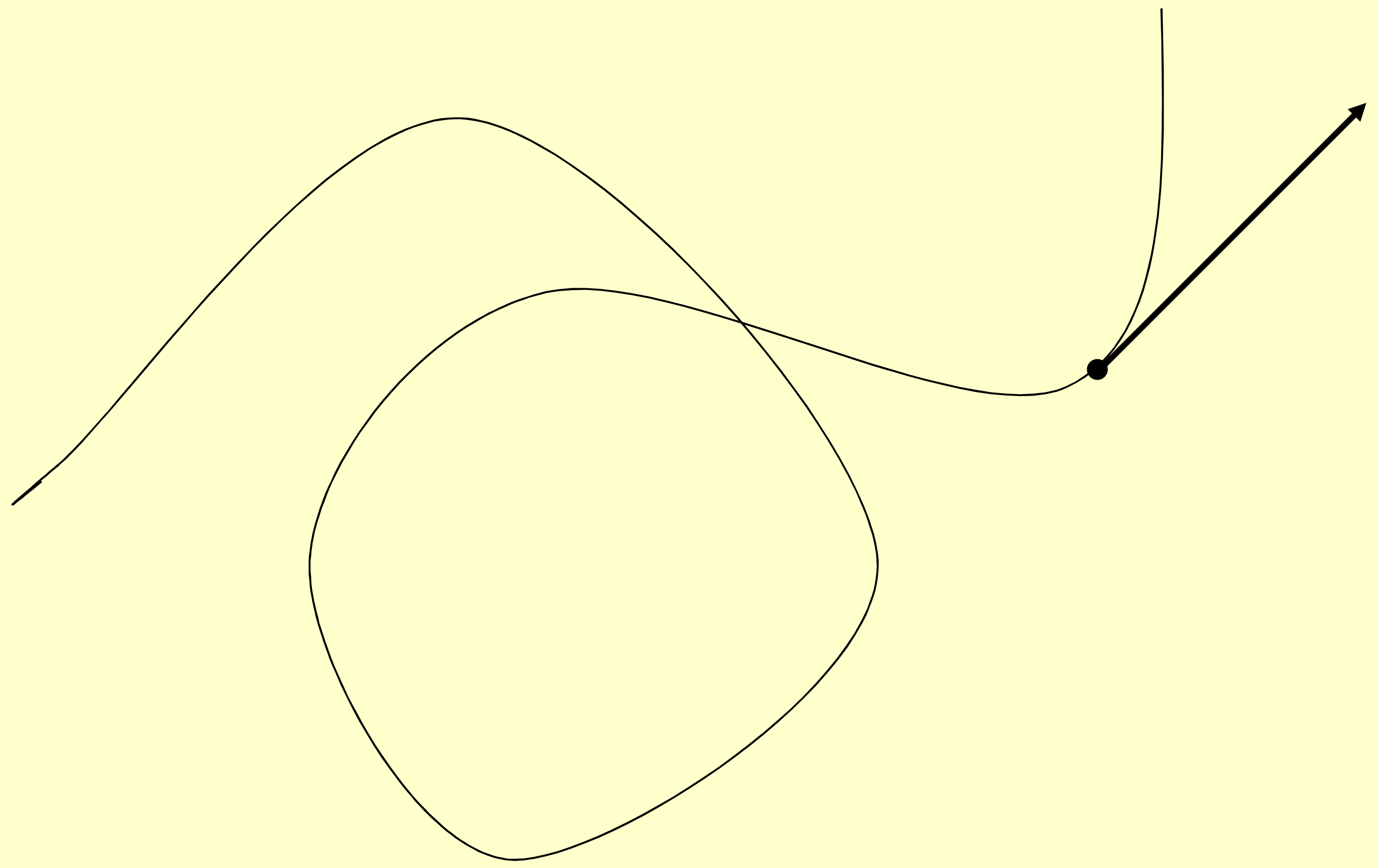
Derivative of a Vector Function

- By definition, the derivative of position is called velocity, and the derivative of velocity is acceleration

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Derivative of a Vector Function



Vector Derivatives

- We’ve seen how to take a derivative of a scalar vs. a scalar, and a vector vs. a scalar
- What about the derivative of a scalar vs. a vector, or a vector vs. a vector?

Vector Derivatives

- Derivatives of scalars with respect to vectors show up often in field equations, used in exciting subjects like fluid dynamics, solid mechanics, and other physically based animation techniques. If we are lucky, we'll have time to look at these later in the quarter
- Today, however, we will be looking at derivatives of vector quantities with respect to other vector quantities

Jacobians

- A Jacobian is a vector derivative with respect to another vector
- If we have a vector valued function of a vector of variables **f**(**x**), the Jacobian is a matrix of partial derivatives- one partial derivative for each combination of components of the vectors
- The Jacobian matrix contains all of the information necessary to relate a change in any component of **x** to a change in any component of **f**
- The Jacobian is usually written as J(**f**,**x**), but you can really just think of it as d**f**/d**x**

Jacobians

$$J(\mathbf{f},\mathbf{x}) = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \cdots & \frac{\partial f_M}{\partial x_N} \end{bmatrix}$$

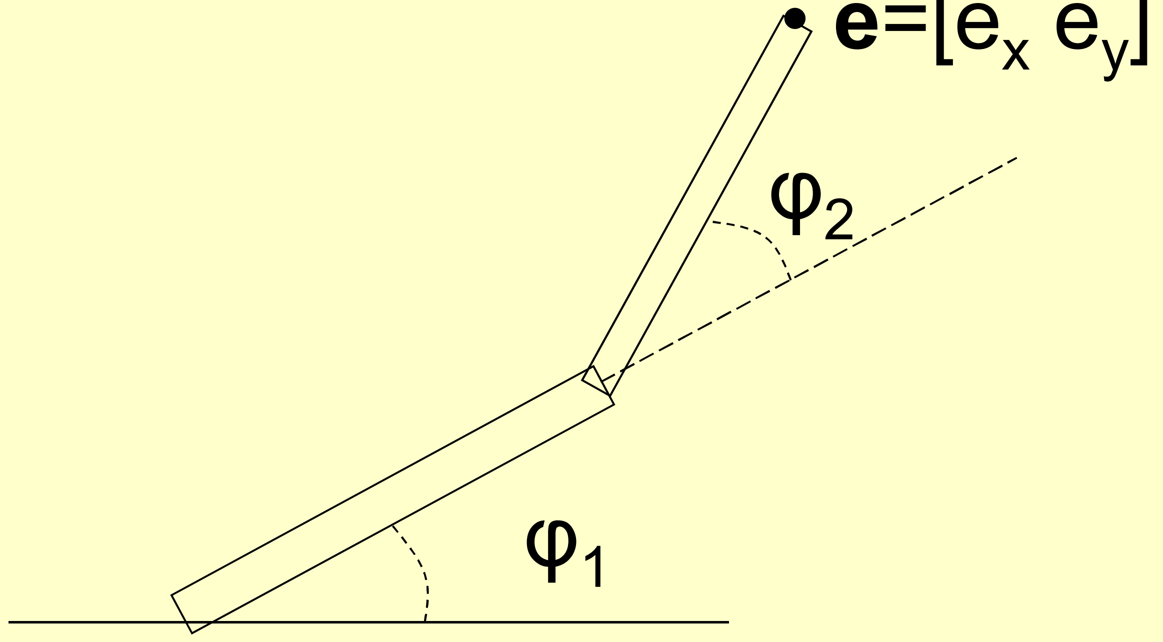
Partial Derivatives

- The use of the ∂ symbol instead of d for partial derivatives really just implies that it is a single component in a vector derivative
- For *many* practical purposes, an individual partial derivative behaves like the derivative of a scalar with respect to another scalar

Jacobian Inverse Kinematics

Jacobians

- Let's say we have a simple 2D robot arm with two 1-DOF rotational joints:



Jacobians

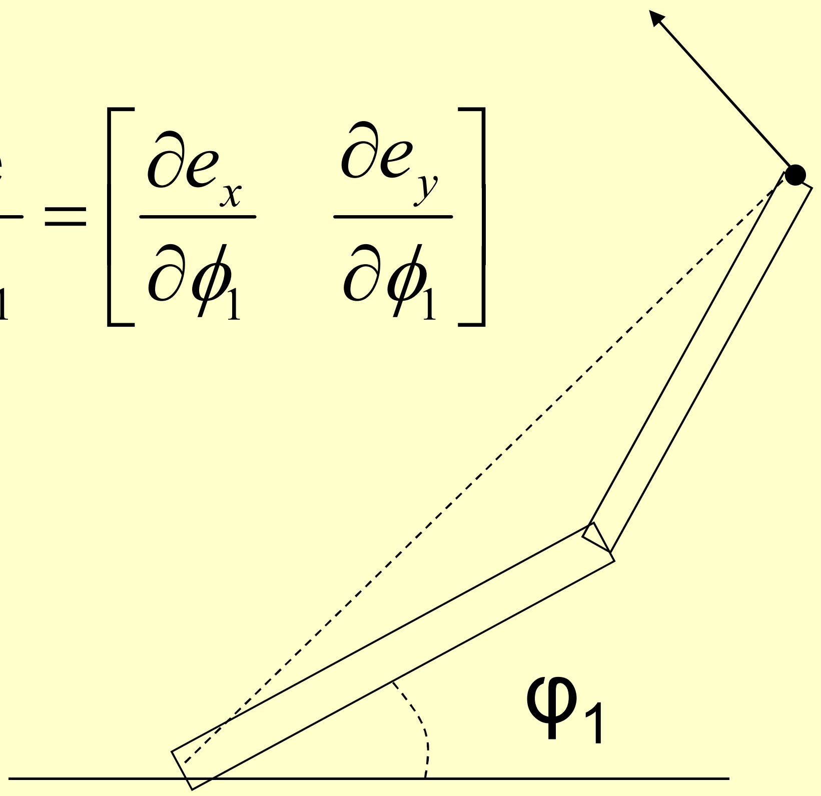
- The Jacobian matrix J(**e**,**Φ**) shows how each component of **e** varies with respect to each joint angle

$$J(\mathbf{e},\mathbf{\Phi}) = \begin{bmatrix} \dfrac{\partial e_x}{\partial \phi_1} & \dfrac{\partial e_x}{\partial \phi_2} \\ \dfrac{\partial e_y}{\partial \phi_1} & \dfrac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$

Jacobians

- Consider what would happen if we increased φ_1 by a small amount. What would happen to **e** ?

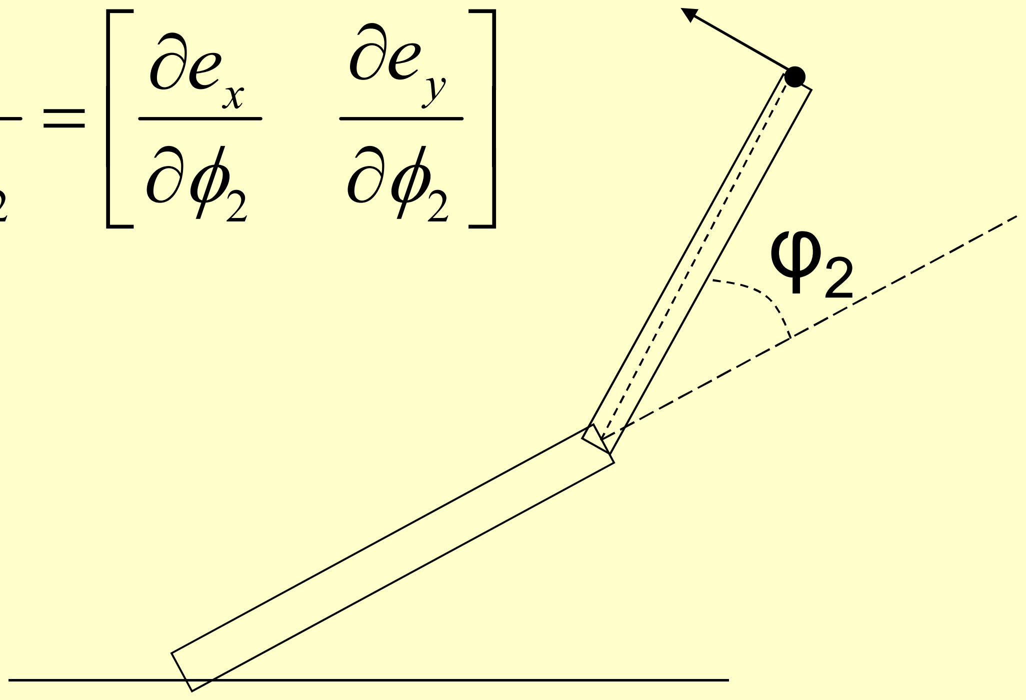
$$\frac{\partial \mathbf{e}}{\partial \phi_1} = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_1} & \frac{\partial e_y}{\partial \phi_1} \end{bmatrix}$$



Jacobians

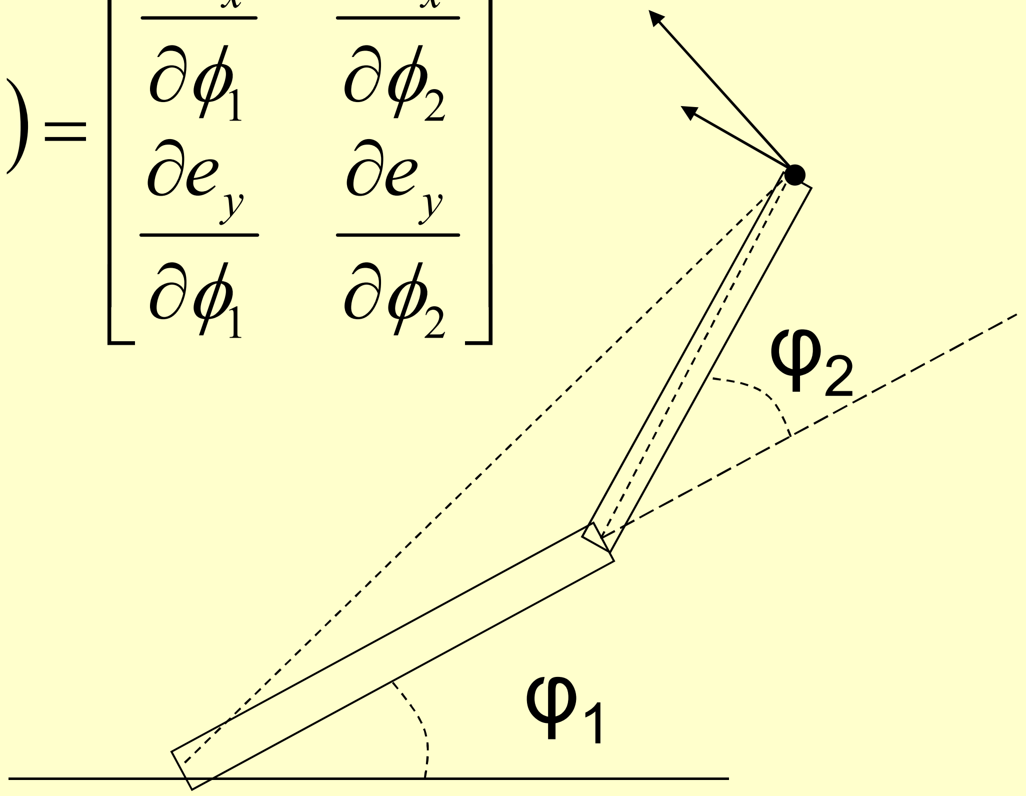
- What if we increased φ_2 by a small amount?

$$\frac{\partial \mathbf{e}}{\partial \phi_2} = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_2} & \frac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$



Jacobian for a 2D Robot Arm

$$J(\mathbf{e},\mathbf{\Phi}) = \begin{bmatrix} \dfrac{\partial e_x}{\partial \phi_1} & \dfrac{\partial e_x}{\partial \phi_2} \\ \dfrac{\partial e_y}{\partial \phi_1} & \dfrac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$



Jacobian Matrices

- Just as a scalar derivative df/dx of a function f(x) can vary over the domain of possible values for x, the Jacobian matrix J(**e**,**Φ**) varies over the domain of all possible poses for **Φ**
- For any given joint pose vector **Φ**, we can explicitly compute the individual components of the Jacobian matrix

Jacobian as a Vector Derivative

- Once again, sometimes it helps to think of:

$$J(\mathbf{e}, \mathbf{\Phi}) = \frac{d\mathbf{e}}{d\mathbf{\Phi}}$$

because J(**e**,**Φ**) contains all the information we need to know about how to relate changes in any component of **Φ** to changes in any component of **e**

Incremental Change in Pose

- Lets say we have a vector **ΔΦ** that represents a small change in joint DOF values
- We can approximate what the resulting change in **e** would be:

$$\Delta\mathbf{e} \approx \frac{d\mathbf{e}}{d\mathbf{\Phi}} \cdot \Delta\mathbf{\Phi} = J(\mathbf{e}, \mathbf{\Phi}) \cdot \Delta\mathbf{\Phi} = \mathbf{J} \cdot \Delta\mathbf{\Phi}$$

Incremental Change in Effector

- What if we wanted to move the end effector by a small amount Δ**e**. What small change Δ**Φ** will achieve this?

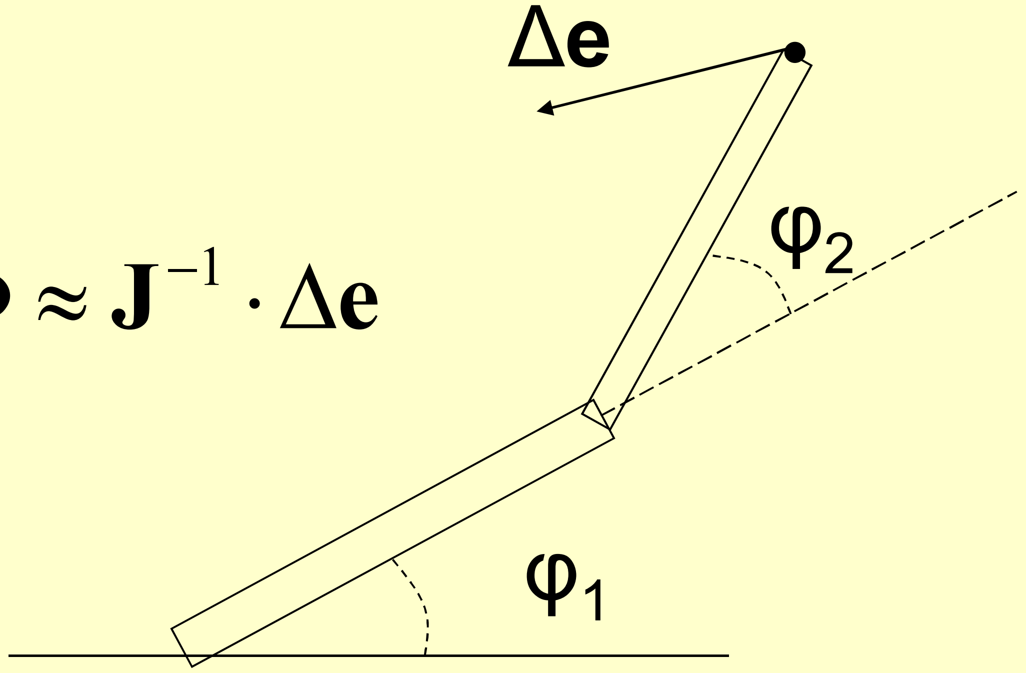
$$\Delta\mathbf{e} \approx \mathbf{J} \cdot \Delta\mathbf{\Phi}$$

$$so:$$

$$\Delta\mathbf{\Phi} \approx \mathbf{J}^{-1} \cdot \Delta\mathbf{e}$$

Incremental Change in **e**

- Given some desired incremental change in end effector configuration Δ**e**, we can compute an appropriate incremental change in joint DOFs Δ**Φ**



$$\Delta\mathbf{\Phi} \approx \mathbf{J}^{-1} \cdot \Delta\mathbf{e}$$

Incremental Changes

- Remember that forward kinematics is a nonlinear function (as it involves sin’s and cos’s of the input variables)
- This implies that we can only use the Jacobian as an approximation that is valid near the current configuration
- Therefore, we must repeat the process of computing a Jacobian and then taking a small step towards the goal until we get to where we want to be

End Effector Goals

- If **Φ** represents the current set of joint DOFs and **e** represents the current end effector DOFs, we will use **g** to represent the goal DOFs that we want the end effector to reach

Choosing $\Delta \mathbf{e}$

- We want to choose a value for $\Delta \mathbf{e}$ that will move $\mathbf{e}$ closer to $\mathbf{g}$. A reasonable place to start is with

$$\Delta \mathbf{e} = \mathbf{g} - \mathbf{e}$$

- We would hope then, that the corresponding value of $\Delta \boldsymbol{\Phi}$ would bring the end effector exactly to the goal
- Unfortunately, the nonlinearity prevents this from happening, but it should get us closer
- Also, for safety, we will take smaller steps:

$$\Delta \mathbf{e} = \beta(\mathbf{g} - \mathbf{e})$$

where $0 \leq \beta \leq 1$

Basic Jacobian IK Technique

```
while ($\mathbf{e}$ is too far from $\mathbf{g}$) {
        Compute $J(\mathbf{e},\mathbf{\Phi})$ for the current pose $\mathbf{\Phi}$
        Compute $J^{-1}$                // invert the Jacobian matrix
        $\Delta\mathbf{e} = \beta(\mathbf{g} - \mathbf{e})$              // pick approximate step to take
        $\Delta\mathbf{\Phi} = J^{-1} \cdot \Delta\mathbf{e}$            // compute change in joint DOFs
        $\mathbf{\Phi} = \mathbf{\Phi} + \Delta\mathbf{\Phi}$            // apply change to DOFs
        Compute new $\mathbf{e}$ vector // apply forward
                                        // kinematics to see
                                        // where we ended up
}
```

A Few Questions

- How do we compute **J** ?
- How do we invert **J** to compute $\mathbf{J}^{-1}$?
- How do we choose β (step size)
- How do we determine when to stop the iteration?

Computing the Jacobian

Computing the Jacobian Matrix

- We can take a geometric approach to computing the Jacobian matrix
- Rather than look at it in 2D, let’s just go straight to 3D
- Let’s say we are just concerned with the end effector position for now. Therefore, **e** is just a 3D vector representing the end effector position in world space. This also implies that the Jacobian will be an 3xN matrix where N is the number of DOFs
- For each joint DOF, we analyze how **e** would change if the DOF changed

1-DOF Rotational Joints

- We will first consider DOFs that represents a rotation around a single axis (1-DOF hinge joint)
- We want to know how the world space position **e** will change if we rotate around the axis. Therefore, we will need to find the axis and the pivot point in world space
- Let’s say φ_i represents a rotational DOF of a joint. We also have the offset $\mathbf{r}_i$ of that joint relative to it’s parent and we have the rotation axis $\mathbf{a}_i$ relative to the parent as well
- We can find the world space offset and axis by transforming them by their parent joint’s world matrix

1-DOF Rotational Joints

- To find the pivot point and axis in world space:

$$\mathbf{a}_i' = \mathbf{a}_i \cdot \mathbf{W}_{i-parent}$$

$$\mathbf{r}_i' = \mathbf{r}_i \cdot \mathbf{W}_{i-parent}$$

- Remember these transform as homogeneous vectors. **r** transforms as a position [r_x r_y r_z 1] and **a** transforms as a direction [a_x a_y a_z 0]

Rotational DOFs

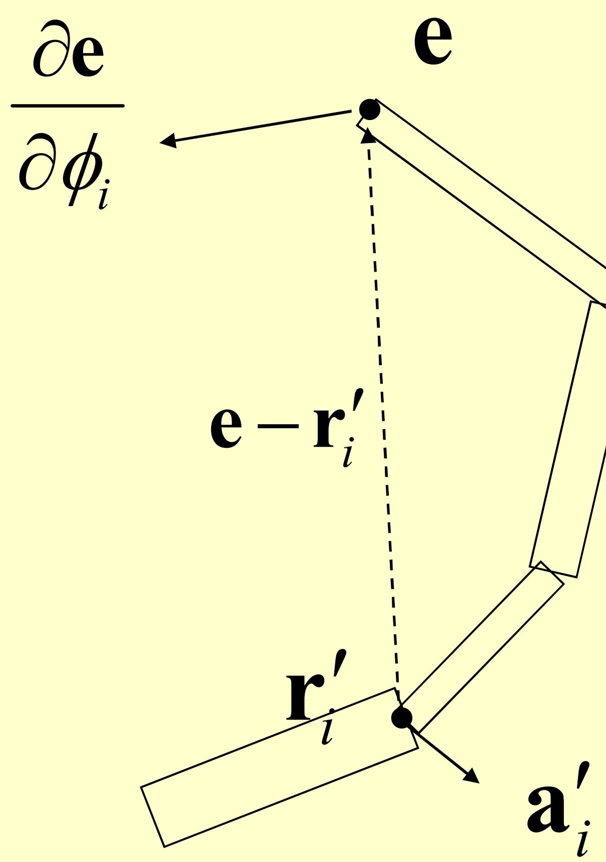
- Now that we have the axis and pivot point of the joint in world space, we can use them to find how **e** would change if we rotated around that axis

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i \times \left(\mathbf{e} - \mathbf{r}'_i\right)$$

- This gives us a column in the Jacobian matrix

Rotational DOFs

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i \times (\mathbf{e} - \mathbf{r}'_i)$$



$\frac{\partial \mathbf{e}}{\partial \phi_i}$
$\mathbf{e}$
$\mathbf{e} - \mathbf{r}'_i$
$\mathbf{r}'_i$
$\mathbf{a}'_i$

a’$_i$: unit length rotation axis in world space

r’$_i$: position of joint pivot in world space

e: end effector position in world space

3-DOF Rotational Joints

- For a 2-DOF or 3-DOF joint, it is actually a little trickier to get the world space axis
- Consider how we would find the world space x-axis of a 3-DOF ball joint
- Not only do we need to consider the parent's world matrix, but we need to include the rotation around the next two axes (y and z-axis) as well
- This is because those following rotations will rotate the first axis itself

3-DOF Rotational Joints

- For example, assuming we have a 3-DOF ball joint that rotates in XYZ order:

$$
\begin{aligned}
x-dof: &\quad \mathbf{a}_i' = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \cdot \mathbf{R}_y(\theta_y) \cdot \mathbf{R}_z(\theta_z) \cdot \mathbf{W}_{parent} \\
y-dof: &\quad \mathbf{a}_i' = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \cdot \mathbf{R}_z(\theta_z) \cdot \mathbf{W}_{parent} \\
z-dof: &\quad \mathbf{a}_i' = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{W}_{parent}
\end{aligned}
$$

- Where $R_y(\theta_y)$ and $R_z(\theta_z)$ are y and z rotation matrices

3-DOF Rotational Joints

- Remember that a 3-DOF XYZ ball joint's local matrix will look something like this:

$$\mathbf{L}\left(\theta_x,\theta_y,\theta_z\right)=\mathbf{R}_x\left(\theta_x\right)\cdot\mathbf{R}_y\left(\theta_y\right)\cdot\mathbf{R}_z\left(\theta_z\right)\cdot\mathbf{T}(\mathbf{r})$$

- Where $R_x(\theta_x)$, $R_y(\theta_y)$, and $R_z(\theta_z)$ are x, y, and z rotation matrices, and T(**r**) is a translation by the (constant) joint offset
- So it's world matrix looks like this:

$$\mathbf{W}=\mathbf{R}_x\left(\theta_x\right)\cdot\mathbf{R}_y\left(\theta_y\right)\cdot\mathbf{R}_z\left(\theta_z\right)\cdot\mathbf{T}(\mathbf{r})\cdot\mathbf{W}_{parent}$$

3-DOF Rotational Joints

- Once we have each axis in world space, each one will get a column in the Jacobian matrix
- At this point, it is essentially handled as three 1-DOF joints, so we can use the same formula for computing the derivative as we did earlier:

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i \times \left(\mathbf{e} - \mathbf{r}'_i\right)$$

- We repeat this for each of the three axes

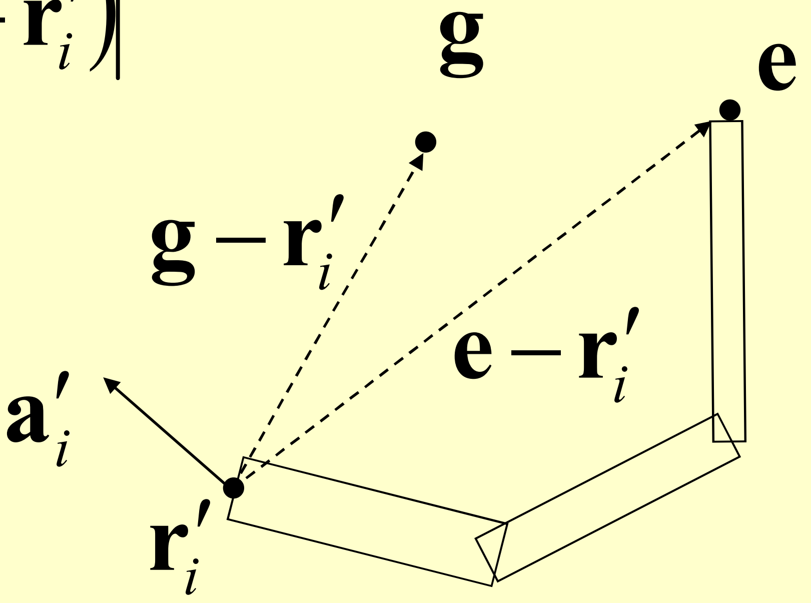
Quaternion Joints

- What about a quaternion joint? How do we incorporate them into our IK formulation?
- We will assume that a quaternion joint is capable of rotating around any axis
- However, since we are trying to find a way to move **e** towards **g**, we should pick the best possible axis for achieving this

$$\mathbf{a}_i' = \frac{(\mathbf{e} - \mathbf{r}_i') \times (\mathbf{g} - \mathbf{r}_i')}{\left|(\mathbf{e} - \mathbf{r}_i') \times (\mathbf{g} - \mathbf{r}_i')\right|}$$

Quaternion Joints

$$\mathbf{a}_i' = \frac{(\mathbf{e}-\mathbf{r}_i')\times(\mathbf{g}-\mathbf{r}_i')}{\left|(\mathbf{e}-\mathbf{r}_i')\times(\mathbf{g}-\mathbf{r}_i')\right|}$$



Quaternion Joints

- We compute **a**$_i$' directly in world space, so we don't need to transform it
- Now that we have **a**$_i$', we can just compute the derivative the same way we would do with any other rotational axis

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i \times \left(\mathbf{e} - \mathbf{r}'_i\right)$$

- We must remember what axis we use, so that later, when we've computed $\Delta\varphi_i$, we know how to update the quaternion

Translational DOFs

- For translational DOFs, we start in the same way, namely by finding the translation axis in world space
- If we had a prismatic joint (1-DOF translation) that could translate along an arbitrary axis $\mathbf{a}_i$ defined in the parent's space, we can use:

$$\mathbf{a}_i' = \mathbf{a}_i \cdot \mathbf{W}_{i-parent}$$

Translational DOFs

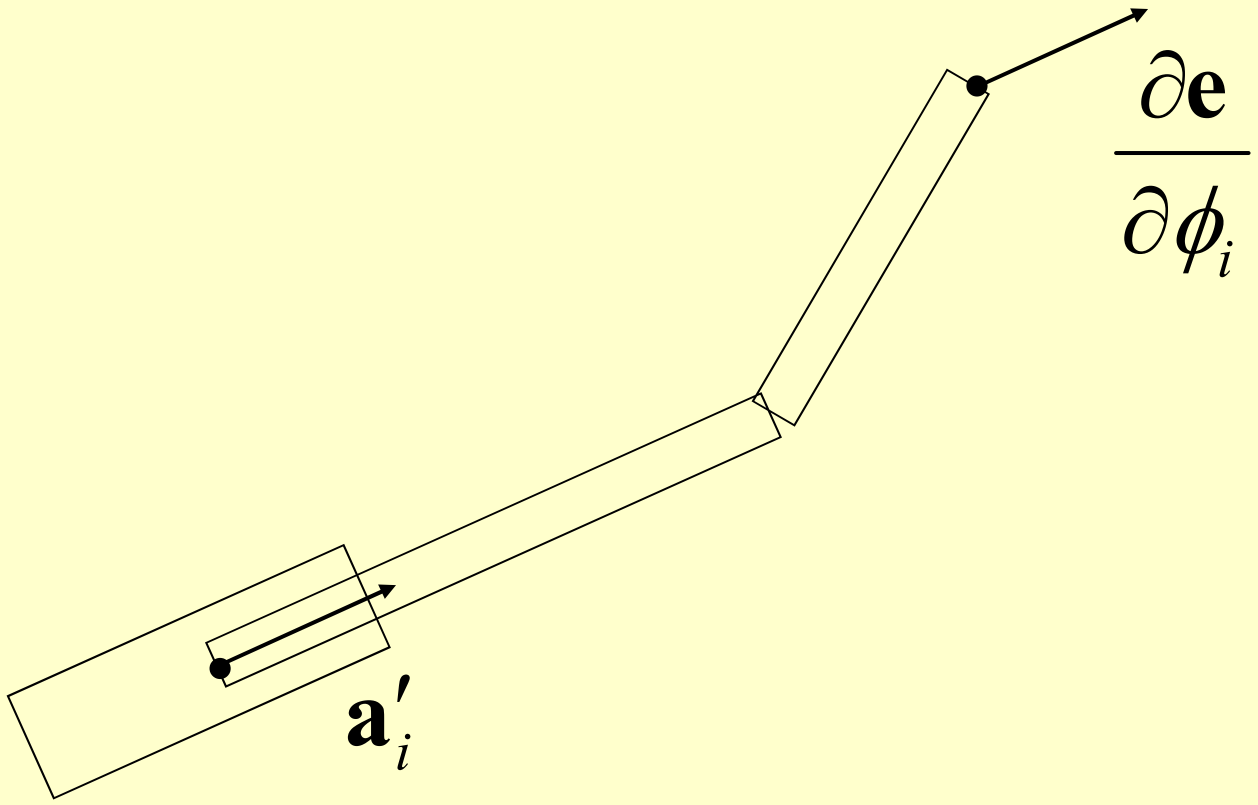
- For a more general 3-DOF translational joint that just translates along the local x, y, and z-axes, we don’t need to do the same thing that we did for rotation
- The reason is that for translations, a change in one axis doesn’t affect the other axes at all, so we can just use the same formula and plug in the x, y, and z axes [1 0 0 0], [0 1 0 0], [0 0 1 0] to get the 3 world space axes
- Note: this will just return the **a**, **b**, and **c** axes of the parent’s world space matrix, and so we don’t actually have to compute them!

Translational DOFs

- As with rotation, each translational DOF is still treated separately and gets its own column in the Jacobian matrix
- A change in the DOF value results in a simple translation along the world space axis, making the computation trivial:

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i$$

Translational DOFs



Building the Jacobian

- To build the entire Jacobian matrix, we just loop through each DOF and compute a corresponding column in the matrix
- If we wanted, we could use more elaborate joint types (scaling, translation along a path, shearing…) and still compute an appropriate derivative
- If absolutely necessary, we could always resort to computing a numerical approximation to the derivative

Units & Scaling

- What about units?
- Rotational DOFs use radians and translational DOFs use meters (or some other measure of distance)
- How can we combine their derivatives into the same matrix?
- Well, it’s really a bit of a hack, but we just combine them anyway
- If desired, we can scale any column to adjust how much the IK will favor using that DOF

Units & Scaling

- For example, we could scale all rotations by some constant that causes the IK to behave how we would like
- Also, we could use this as an additional way to get control over the behavior of the IK
- We can store an additional parameter for each DOF that defines how ‘stiff’ it should behave
- If we scale the derivative larger (but preserve direction), the solution will compensate with a smaller value for $\Delta\varphi i$, therefore making it act stiff
- There are several proposed methods for automatically setting the stiffness to a reasonable default value. They generally work based on some function of the length of the actual bone. The Welman paper talks about this.

End Effector Orientation

End Effector Orientation

- We've examined how to form the columns of a Jacobian matrix for a position end effector with 3 DOFs
- How do we incorporate orientation of the end effector?
- We will add more DOFs to the end effector vector **e**
- Which method should we use to represent the orientation? (Euler angles? Quaternions?…)
- Actually, a popular method is to use the 3 DOF scaled axis representation!

Scaled Rotation Axis

- We learned that any orientation can be represented as a single rotation around some axis
- Therefore, we can store an orientation as an 3D vector
  - The direction of the vector is the rotation axis
  - The length of the vector is the angle to rotate in radians
- This method has some properties that work well with the Jacobian approach
  - Continuous and consistent
  - No redundancy or extra constraints
  - It’s also a nice method to store incremental changes in rotation

6-DOF End Effector

- If we are concerned about both the position and orientation of the end effector, then our **e** vector should contain 6 numbers
- But remember, we don't actually need the **e** vector, we really just need the Δ**e** vector
- To generate Δ**e**, we compare the current end effector position/orientation (matrix **E**) to the goal position/orientation (matrix **G**)
- The first 3 components of Δ**e** represent the desired change in position: β(**G.d** - **E.d**)
- The next 3 represent a desired change in orientation, which we will express as a scaled axis vector

Desired Change in Orientation

- We want to choose a rotation axis that rotates **E** in to **G**
- We can compute this using some quaternions:

  $\mathbf{M}=\mathbf{E}^{-1}\cdot\mathbf{G}$

  q.FromMatrix(**M**);
- This gives us a quaternion that represents a rotation from **E** to **G**
- To extract out the rotation axis and angle, we just remember that:

$$\mathbf{q}=\begin{bmatrix}\cos\frac{\theta}{2} & a_x\sin\frac{\theta}{2} & a_y\sin\frac{\theta}{2} & a_z\sin\frac{\theta}{2}\end{bmatrix}$$

- We can then scale the final axis by β

End Effector

- So we now can define our goal with a matrix and come up with some desired change in end effector values that will bring us closer to that goal:

$$\Delta\mathbf{e} = \begin{bmatrix} \Delta t_x & \Delta t_y & \Delta t_z & \Delta\theta_x & \Delta\theta_y & \Delta\theta_z \end{bmatrix}^T$$

- We must now compute a Nx6 Jacobian matrix, where each column represents how a particular DOF will affect both the position and orientation of the end effector

Rotational DOFs

- We need to compute additional derivatives that show how the end effector orientation changes with respect to an incremental change in each DOF
- We will use the scaled axis to represent the incremental change
- For a rotational DOF, we first find the rotation axis in world space (as we did earlier)
- Then- we're done! That axis already represents the incremental rotation caused by that DOF
- By default, the length of the axis should be 1, indicating that a change of 1 in the DOF value results in a rotation of 1 radian around the axis. We can scale this by a stiffness value if desired

Rotational DOFs

- The column in the Nx6 Jacobian matrix corresponding to a rotational DOF is:

$$\mathbf{J}_i = \frac{\partial \mathbf{e}}{\partial \phi_i} = \begin{bmatrix} \left[\mathbf{a}'_i \times \left(\mathbf{e}_{pos} - \mathbf{r}'_i\right)\right]^T \\ \left[\mathbf{a}'_i\right]^T \end{bmatrix}$$

- **a**’ is the rotation axis in world space
- **r**’ is the pivot point in world space
- $\mathbf{e}_{\text{pos}}$ is the position of the end effector in world space

Translational DOFs

- Translational DOFs don’t affect the end effector orientation, so their contribution to the derivative of orientation will be [0 0 0]

$$\mathbf{J}_i = \frac{\partial \mathbf{e}}{\partial \phi_i} = \begin{bmatrix} [\mathbf{a}'_i]^T \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Inverse Kinematics (part 2)

CSE169: Computer Animation

Instructor: Steve Rotenberg

UCSD, Winter 2005

Inverting the Jacobian Matrix

Inverting the Jacobian

- If the Jacobian is square (number of joint DOFs equals the number of DOFs in the end effector), then we *might* be able to invert the matrix
- Most likely, it won’t be square, and even if it is, it’s definitely possible that it will be singular and non-invertable
- Even if it is invertable, as the pose vector changes, the properties of the matrix will change and may become singular or near-singular in certain configurations
- The bottom line is that just relying on inverting the matrix is not going to work

Underconstrained Systems

- If the system has more degrees of freedom in the joints than in the end effector, then it is likely that there will be a continuum of redundant solutions (i.e., an infinite number of solutions)
- In this situation, it is said to be underconstrained or redundant
- These should still be solvable, and might not even be too hard to find a solution, but it may be tricky to find a ‘best’ solution

Overconstrained Systems

- If there are more degrees of freedom in the end effector than in the joints, then the system is said to be overconstrained, and it is likely that there will not be any possible solution
- In these situations, we might still want to get as close as possible
- However, in practice, overconstrained systems are not as common, as they are not a very useful way to build an animal or robot (they might still show up in some special cases though)

Well-Constrained Systems

- If the number of DOFs in the end effector equals the number of DOFs in the joints, the system could be well constrained and invertable
- In practice, this will require the joints to be arranged in a way so their axes are not redundant
- This property may vary as the pose changes, and even well-constrained systems may have trouble

Pseudo-Inverse

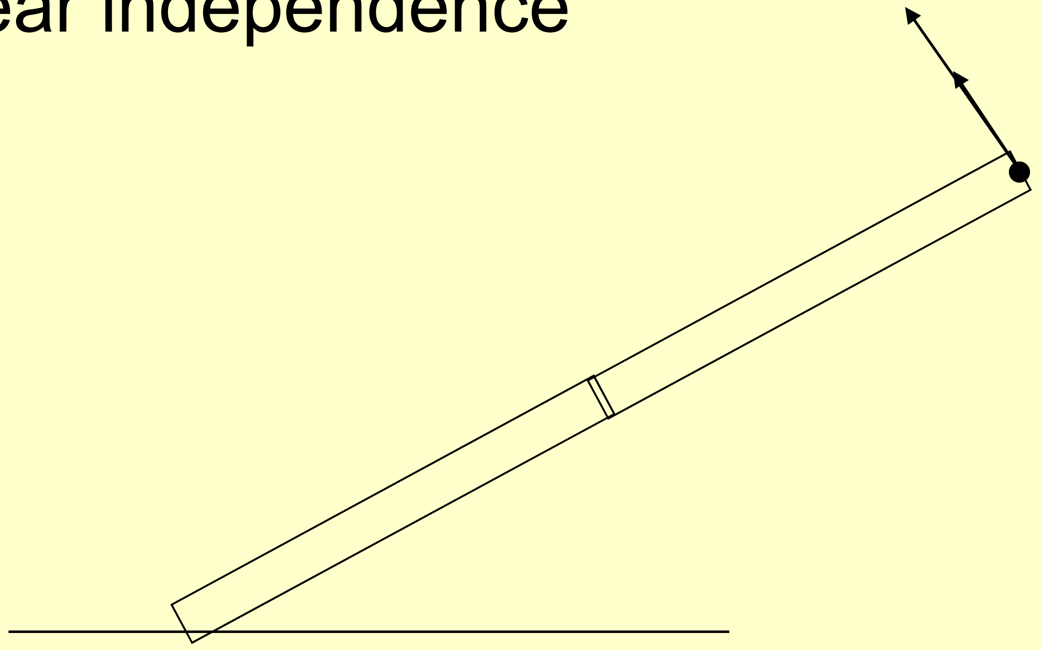
- If we have a non-square matrix arising from an overconstrained or underconstrained system, we can try using the *pseudoinverse*:

$$\mathbf{J^*=(J^TJ)^{-1}J^T}$$

- This is a method for finding a matrix that effectively inverts a non-square matrix

Degenerate Cases

- Occasionally, we will get into a configuration that suffers from degeneracy
- If the derivative vectors line up, they lose their linear independence



Single Value Decomposition

- The SVD is an algorithm that decomposes a matrix into a form whose properties can be analyzed easily
- It allows us to identify when the matrix is singular, near singular, or well formed
- It also tells us about what regions of the multidimensional space are not adequately covered in the singular or near singular configurations
- The bottom line is that it is a more sophisticated, but expensive technique that can be useful both for analyzing the matrix and inverting it

Jacobian Transpose

- Another technique is to simply take the transpose of the Jacobian matrix!
- Surprisingly, this technique actually works pretty well
- It is *much* faster than computing the inverse or pseudo-inverse
- Also, it has the effect of localizing the computations. To compute $\Delta\varphi_i$ for joint i, we compute the column in the Jacobian matrix $\mathbf{J}_i$ as before, and then just use:

$$\Delta\varphi_i = \mathbf{J}_i^T \cdot \Delta e$$

Jacobian Transpose

- With the Jacobian transpose (JT) method, we can just loop through each DOF and compute the change to that DOF directly
- With the inverse (JI) or pseudo-inverse (JP) methods, we must first loop through the DOFs, compute and store the Jacobian, invert (or pseudo-invert) it, then compute the change in DOFs, and then apply the change
- The JT method is far friendlier on memory access & caching, as well as computations
- However, if one prefers quality over performance, the JP method might be better…

Iterating to the Solution

Iteration

- Whether we use the JI, JP, or JT method, we must address the issue of iteration towards the solution
- We should consider how to choose an appropriate step size β and how to decide when the iteration should stop

When to Stop

- There are three main stopping conditions we should account for
  - Finding a successful solution (or close enough)
  - Getting stuck in a condition where we can't improve (local minimum)
  - Taking too long (for interactive systems)
- All three of these are fairly easy to identify by monitoring the progress of **Φ**
- These rules are just coded into the while() statement for the controlling loop

Finding a Successful Solution

- We really just want to get close enough within some tolerance
- If we’re not in a big hurry, we can just iterate until we get within some floating point error range
- Alternately, we could choose to stop when we get within some tolerance measurable in pixels
- For example, we could position an end effector to 0.1 pixel accuracy
- This gives us a scheme that should look good and automatically adapt to spend more time when we are looking at the end effector up close (level-of-detail)

Local Minima

- If we get stuck in a local minimum, we have several options
  - Don’t worry about it and just accept it as the best we can do
  - Switch to a different algorithm (CCD…)
  - Randomize the pose vector slightly (or a lot) and try again
  - Send an error to whatever is controlling the end effector and tell it to try something else
- Basically, there are few options that are truly appealing, as they are likely to cause either an error in the solution or a possible discontinuity in the motion

Taking Too Long

- In a time critical situation, we might just limit the iteration to a maximum number of steps
- Alternately, we could use internal timers to limit it to an actual time in seconds

Iteration Stepping

- Step size
- Stability
- Performance

Other IK Issues

Joint Limits

- A simple and reasonably effective way to handle joint limits is to simply clamp the pose vector as a final step in each iteration
- One can’t compute a proper derivative at the limits, as the function is effectively discontinuous at the boundary
- The derivative going towards the limit will be 0, but coming away from the limit will be non-zero. This leads to an inequality condition, which can’t be handled in a continuous manner
- We could just choose whether to set the derivative to 0 or non-zero based on a reasonable guess as to which way the joint would go. This is easy in the JT method, but can potentially cause trouble in JI or JP

Higher Order Approximation

- The first derivative gives us a linear approximation to the function
- We can also take higher order derivatives and construct higher order approximations to the function
- This is analogous to approximating a function with a Taylor series

Repeatability

- If a given goal vector **g** always generates the same pose vector **Φ**, then the system is said to be repeatable
- This is not likely to be the case for redundant systems unless we specifically try to enforce it
- If we always compute the new pose by starting from the last pose, the system will probably not be repeatable
- If, however, we always reset it to a ‘comfortable’ default pose, then the solution should be repeatable
- One potential problem with this approach however is that it may introduce sharp discontinuities in the solution

Multiple End Effectors

- Remember, that the Jacobian matrix relates each DOF in the skeleton to each scalar value in the **e** vector
- The components of the matrix are based on quantities that are all expressed in world space, and the matrix itself does not contain any actual information about the connectivity of the skeleton
- Therefore, we extend the IK approach to handle tree structures and multiple end effectors without much difficulty
- We simply add more DOFs to the end effector vector to represent the other quantities that we want to constrain
- However, the issue of scaling the derivatives becomes more important as more joints are considered

Multiple Chains

- Another approach to handling tree structures and multiple end effectors is to simply treat it as several individual chains
- This works for characters often, as we can animate the body with a forward kinematic approach, and then animate each limb with IK by positioning the hand/foot as the end effector goal
- This can be faster and simpler, and actually offer a nicer way to control the character

Geometric Constraints

- One can also add more abstract geometric constraints to the system
  - Constrain distances, angles within the skeleton
  - Prevent bones from intersecting each other or the environment
  - Apply different weights to the constraints to signify their importance
  - Have additional controls that try to maximize the 'comfort' of a solution
  - Etc.
- Welman talks about this in section 5

Other IK Techniques

- Cyclic Coordinate Descent
  - This technique is more of a trigonometric approach and is more heuristic. It does, however, tend to converge in fewer iterations than the Jacobian methods, even though each iteration is a bit more expensive. Welman talks about this method in section 4.2
- Analytical Methods
  - For simple chains, one can directly invert the forward kinematic equations to obtain an exact solution. This method can be very fast, very predictable, and precisely controllable. With some finesse, one can even formulate good analytical solvers for more complex chains with multiple DOFs and redundancy
- Other Numerical Methods
  - There are lots of other general purpose numerical methods for solving problems that can be cast into **f**(**x**)=**g** format

Jacobian Method as a Black Box

- The Jacobian methods were not invented for solving IK. They are a far more general purpose technique for solving systems of non-linear equations
- The Jacobian solver itself is a black box that is designed to solve systems that can be expressed as $\mathbf{f}(\mathbf{x})=\mathbf{g}$ $\quad$ ($\mathbf{e}(\mathbf{\Phi})=\mathbf{g}$)
- All we need is a method of evaluating **f** and **J** for a given value of **x** to plug it into the solver
- If we design it this way, we could conceivably swap in different numerical solvers (JI, JP, JT, damped least-squares, conjugate gradient…)