Algorithm Analysis Beyond Time: Communication and Friends

Communication Complexity

1. Alice and Bob get n integers each.
   There is exactly one integer in $[1, 2n+1]$ that does not appear in anyone's input. Every other integer appears once.
   Find the missing number with as little communication as possible.

2. Alice gets $n+1$ integers, Bob gets n integers.
   If we combine their inputs, there is exactly one integer appearing only once, every other integer appears twice (possibly once in Alice's and once in Bob's input; or twice in Alice's and none in Bob's, or vice versa).
   a. Find the special number with as little communication as possible.
   b. Hard: What if there are two special numbers?

3. Consider the communication matrix of Equality.
   a. Show that any monochromatic rectangle of "=" must be of size 1x1.
   b. Use this fact to conclude that Alice and Bob must communicate at least n bits.

4. [Indexing] Alice gets an n-bit string x, and Bob gets an index $i \in \{1, \dots, n\}$.
   Alice sends one message to Bob who must determine the i^{th} bit of x with minimum communication.
   a. Prove that Alice's message must be at least n bits.
   b. If Alice and Bob can talk back and forth, show that they only need $\log_2 n + 1$ bits.

5. [Set-disjointness] Alice and Bob each get a subset of $\{1, .., n\}$.
   They want to determine if their subsets are disjoint (no common element) with minimum communication.
   Prove that they need to send at least n bits.

6. Hard: Suppose there exists a function $f: [2^n]x[2^n] \rightarrow \{0,1\}$ that can be covered with 2^k geometric monochromatic rectangles.
   A geometric monochromatic rectangle is a set of the form
   $$R = \{(x, y) \mid x_{min} \leq x \leq x_{max} \text{ and } y_{min} \leq y \leq y_{max}\}$$
   such that $f(x, y)$ is constant for all $(x, y) \in R$.
   Show that Alice and Bob can compute f using $O(k)$ bits.

Streaming

* For question 6-7, assume that you can memorize a number using 1 bit of space.

7. [Similar to exercise 1] The stream consists of $2n$ integers.
   There is exactly one integer in $[1,2n + 1]$ that does not appear in the stream. Every other integer appears once.
   Find the missing number using $O(1)$ space and 1 pass.

8. [Similar to exercise 2, but harder!] The stream consists of $2n + 1$ integers from $[1, n + 1]$.
   There is exactly one integer that appears once, every other integer appears twice.
   a. Find the special number using $O(1)$ space and 1 pass.
   b. If there are 2 special numbers, can you use $O(1)$ space and 2 passes?
   c. Use the solution to exercise 8.b to solve exercise 2.b.
   d. Harder: If there are 2 special numbers, can you use $O(1)$ space and 1 pass?

9. Let A be subsets of $\{1, . . , n\}$. The stream first contains the numbers of B, then a special symbol $, and then the numbers of B.
   We want to determine if A and B are disjoint (no common element) in 1 pass.
   Use the result of exercise 4 to show a lower bound of n bits on the required space.
   Hint: given a streaming algorithm using space S, how to turn it into a communication protocol where Alice sends S bits to Bob?